NASA TT F-9700

NASA TT F-9700

HYPERGEOMETRIC FUNCTIONS AND VARIOUS RELATED PROBLEMS

S. Pincherle

Translation of "Delle funzioni ipergeometriche
e di varie questioni ad esse attinenti."
Giornali di Matematiche de Battaglini,
Vol. 32, pp. 209-291, Naples 1894.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON DECEMBER 1965

NASA TT F-9700

HYPERGEOMETRIC FUNCTIONS AND VARIOUS RELATED PROBLEMS

S. Pincherle

The author presents a series of lessons dealing with the theory of hypergeometric functions based on the principles of analytic function theory. Then dealt with successively are the principal properties of hypergeometric series, linear difference equation and second order linear difference equation, linear differential equations and regular linear differential equations and their application to the hypergeometric equation. */209

Chapter VI contains the theory of a quite simple functional transformation, and, as an application of this operation, two distinct generalizations of hypergeometric functions by Pochhammer and Goursat. In Chapter VII, by applying some general propositions in linear differential equations, particularly of the second order, the following results, among others, are given: 1. a method of calculating the value of a continuous fraction the terms of which are rational functions of the index and, as a special case, the well-known Gauss formula for the expansion of the quotient of two contiguous hypergeometric series in a continuous fraction; 2. the development of an analytic function in ordinate series according to the denominators or the remainders of the reductions of a continuous algebraic fraction, especially in accordance with a system of hypergeometric functions, e.g., spherical functions, with a method for determining the convergence terms.

The hypergeometric series, studied by Gauss as a synthesis of elementary trascendentals, has in turn been taken as a point of departure in the research and formation of innumerable classes of new functions. The important part played by the Gauss series in the development of modern analytical methods is evident if one takes into consideration the fact that two of the most outstanding theories which have enriched science in the last thirty years have resulted from studies on this series: that of linear differential equations established by Fuchs in a now classic memorandum, the elementary origins of which, however,

*/Numbers in the margin indicate pagination of the original foreign text.

are found in the celebrated work by Riemann on the hypergeometric series, and that of automorphic functions, which owes its results to Poincaré and Klein, the method of which, however, appears to be contained in essence in the studies by Schwarz on cases in which the hypergeometric differential equation admits of an algebraic integral. Because of their historical interest and the number of generalizations which, even in recent times, the various mathematicians have given them in different directions, and due to the various useful digressions which it offers, I was of the opinion that the theory of hypergeometric functions would be an admirable subject for a series of lessons for students adequately grounded in the principles of analytical function theory.

Therefore, I have presented a series of lessons on this subject in the R. University of Bologna during the current school year. In gathering material for this course I have noticed that various theories, presented as separate generalizations of the hypergeometric functions, can instead arise from a single source and that some treatments, with no apparent connection with each other, presented at different times and with different methods, could be regrouped under a single point of view, which is doubly advantageous in that /210 greater simplicity and brevity of exposition and uniformity of method are achieved. This observation has led me to believe that the publication of portions of this course would serve a useful purpose; I was moved to this end also by the consideration that this publication would give me the opportunity to present the applications of some results obtained by me in previous works in a simplified form. Chapter VI of this memorandum therefore contains the theory of a quite simple operation or transformation which I previously encountered; in the same chapter, as an application of this operation, are presented in a very obvious manner two distinct generalizations of the hypergeometric functions which, originating from quite different views, have been given by Pochhammer and Goursat. Also, in Chapter VII, by applying some general propositions on linear differential equations, particularly of the second order, the following results, among others, are given: 1. a method of calculating the value of a continuous fraction the terms of which are rational functions of the index and, as a special case, the well-known Gauss formula for the expansion of the quotient of two contiguous hypergeometric series in a continuous fraction; 2. the development of an analytic function in ordinate series according to the denominators or the remainders of the reductions of a continuous algebraic fraction, especially in accordance with a system of hypergeometric functions, e.g., spherical functions, with a method for determining the convergence conditions (completely different from that followed by Thomé in his memorandum of Vol. 66 of the Journal of Crelle) to which I should like to call the attention of the reader because I believe that it can be found in a more simple and much more general manner.

In view of the fact that the journal in which I have proposed to publish these few pages is directed to the young students in the Italian universities by its illustrious and lamented founder and that the subject involved is of particular interest to them, I felt that it was my duty to present the material in the most accessible form and to hold the necessary acquirements to a minimum. Therefore, I feel it necessary to touch on some well-known facts: these are found in the first chapter, in which the most obvious properties of the Gauss series are simply summarized, in the third chapter, which contains the elements of the theory of linear differential equations, and partly in the second chapter, in which the theory of recurring (or periodic) linear (or difference) equations particularly of the second order is developed with a certain degree

of amplitude. The usefulness of the latter theory is evident on each page of the following chapters. It may seem superfluous to present a theory as well known as that of linear differential equations. I decided to do so for a number of reasons: first, the continuous references made to it in the subsequent work; second, the fact that this theory is less familiar to our students than it should be due to the fact that we do not possess a textbook which presents it clearly and simply and that it is difficult for many students to read the original memorandums; finally, because the opportunity of adapting a most genial method of the lamented Prof. Casorati to school use presents itself here, a method which consists of adding to the exposition of the theory of /211
linear differential equations, the concepts on linear difference equations and which offers unparalleled simplicity and scientific as well as didactic interest.

It would be fortunate if my endeavor would induce others with greater ability to compile a complete work on the generalization of the hypergeometric functions which would contain the following data conveniently fused together in the common point of view of the theory of substitution groups: the results obtained by Schwarz* in seeking algebraic solutions for the hypergeometric differential equation, by Heun** by increasing the number of its singular points, by Papperitz***, who made a contribution to the study of the uniform, automorphic functions, which originate from the equation itself, by Klein****, who so

---

*Crelle, Vol. 65, p. 292.

**Math. Annalen, Vol. 33, p. 161.

***Math. Annalen, Vol. 34, p. 247.

****Math. Annalen, Vol. 37, p. 573.

splendidly discovered the roots of the hypergeometric series, and by many others*, not to mention the extension of the hypergeometric functions to the case of many variables, already successfully attempted by Picard**, by Appell*** and by Horn****, and which by itself would offer sufficient material for a separate monograph.

CHAPTER I

SUMMARY OF THE PRINCIPAL PROPERTIES OF THE HYPERGEOMETRIC SERIES

1. From the term of <u>progression</u> or <u>geometric series</u> given to the expansion:

$$1 + x + x^2 + \ldots. + x^n + \ldots.$$

comes the term <u>hypergeometric series</u> given by Euler to the series:

$$1 + \frac{\alpha\cdot\beta}{\gamma\cdot 1}x + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1\cdot 2}x^2 + \ldots + \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\cdot\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)\cdot 1\cdot 2\cdot 3\ldots n}x^n + \ldots \qquad (1)$$

Gauss, in a now classic work[V], has made a thorough study of this series in- /212 sofar as was possible with the analytical knowledge of his time; he designated the series (1) with the symbol

$$F(\alpha, \beta, \gamma, x)$$

---

*As the present memorandum was in the final stages of composition, I learned that the most distinguished Prof. Klein has presented a course on hypergeometric functions, now published in lithographic form, during the past winter semester. As yet, I have not been able to secure information on that course, but surely the "desideratum" here formulated will be well rewarded.

**Annales de l'École Normale Supérieure, 1881

***J. Math., S. III, Vol. 8, p. 173.

****Acta Mathematica, Vol. 15, p. 113.

[V]Werke, Vol. III, p. 123.

calling α, β, and γ parameters and x the argument: notations and denominations which have been retained by his successors.

The series (1) contains as special cases many of the series which are presented in the elements of calculus. We shall point out the following:

$$F(-m, \beta, \beta, -x) = \sum_{0}^{\infty} \binom{m}{n} x^n = (1+x)^m, \tag{a}$$

i.e., the binominal series;

$$F(1, 1, 2, -x) = \sum_{0}^{\infty} (-1)^n \frac{x^n}{n+1} = \frac{1}{x} \log(1+x), \tag{b}$$

from which, the logarithmic series; etc.

We should also note, more for its historical interest than for its scientific significance, that

$$F\left(1, \rho, 1, \frac{x}{\rho}\right) = \sum_{0}^{\infty} \frac{\rho(\rho+1)\dots(\rho+n-1)}{n!\,\rho^n} x^n$$

yields at the limit for $\rho = \infty$, the function e^x, as shown by a very obvious reasoning which is performed by dividing the summation into two parts, the first of which contains a number of terms independent of ρ. In a similar way it is also found that

$$F\left(\rho, \rho', \frac{1}{2}, -\frac{x^2}{\rho\rho'}\right)$$

is reduced when $\rho = \infty$, $\rho' = \infty$, to the series expansion of cos x. However, we shall not consider infinite values of the parameters in the following.

2. If a ratio is established in the series (1) between a term and the preceding one, it is immediately evident that the limit of this ratio for /213

$n = \infty$ is x: therefore, it can be concluded that the series is absolutely convergent for $|x| < 1$, and divergent for $|x| > 1$. In the plane of the complex variable x the series (1) then has a circle of convergence with the center at $x = 0$ and with the radius equal to unity: this series then has a regular analytical function at every point within this circle and thus possesses all the properties, from the principles of the theory to those of the analytical functions, which can be expected with these functions in the intervals in which they remain regular.

For $|x| = 1$, the criterion used permits of doubt with respect to the convergence or divergence of the series. It is therefore readily demonstrable that the series converges absolutely also for $|x| = 1$ when the real part of $\alpha + \beta - \gamma$ is negative. Indeed, by putting

$$\alpha = \alpha' + i\alpha'' \quad , \quad \beta = \beta' + i\beta'' \quad , \quad \gamma = \gamma' + i\gamma''$$

we obtain easily that

$$\left| \frac{(\alpha + n)(\beta + n)}{(\gamma + n)(n + 1)} \right| = 1 + \frac{\alpha' + \beta' - \gamma' - 1}{n} + \text{superior powers of } \frac{1}{n}.$$

Keeping in mind that the series of positive terms Σa_n is convergent if

$$\lim_{n=\infty} n\left(1 - \frac{a_n}{a_{n-1}}\right) > 1 ,$$

it can be concluded that when $\alpha' + \beta' - \gamma'$ is negative, the series

$$1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1) 1 \cdot 2} x^2 + \dots$$

is convergent absolutely for $|x| = 1$.

The hypergeometric series is reduced to a polynomial if, and only if, one of the α, β numbers is a negative integer.

3. If α is a complex number, ρ a positive number, and $|\alpha| \leq \rho$, we have

$$|\alpha(\alpha+1)\ldots(\alpha+n-1)| \leq \rho(\rho+1)\ldots(\rho+n-1),$$

therefore, for the x values located on the circumference of center x = 0 and of radius $\xi < 1$ and for α values within a circle of center $\alpha = 0$ and of radius ρ, we have

$$|F(\alpha, \beta, \gamma, x)| \leq |F(\rho, \beta, \gamma, \xi)|.$$

If M is now the upper limit of the second member values, for a known theorem /214 on the power series, we have

$$\left|\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)\,1\cdot 2\cdot 3\ldots n}\right| \leq \frac{M}{\xi^n},$$

and this is sufficient reason to state that, for the given values of α and $\xi' < \xi$, the series $F(\alpha, \beta, \gamma, \xi')$ is equally convergent. A fundamental theorem of Weierstrass on the series of rational functions* now permits us to conclude that the said series is a uniform analytical function of α, regular for all finite α values, i.e., that this series is an <u>integral trascendental</u> function of α. The same is true in cases of F considered as a function of β.

Now let us consider F as a function of γ. If $\gamma = \gamma' + i\gamma''$ is still true and we assume that $\gamma' + m > 0$, m being a positive integral number, we shall have

$$|\gamma + m| \geq \gamma' + m,$$

therefore

$$\left|\frac{1}{(\gamma+m)(\gamma+m+1)\ldots(\gamma+n-1)}\right| \leq \frac{1}{(\gamma'+m)(\gamma'+m+1)\ldots(\gamma'+n-1)}.$$

---

*Monatsberichte der Akad. der Wissensch. zu Berlin, August, 1880.

Thus the series F can be written

$$F = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} x + \ldots + \frac{\alpha(\alpha+1)\ldots(\alpha+m-1)\,\beta(\beta+1)\ldots(\beta+m-1)}{\gamma(\gamma+1)\ldots(\gamma+m-1)\,1\cdot 2\cdot 3\ldots m} x^m F_1 ,$$

where, given $|x| = \xi$ and γ''' being a positive number less than $\gamma' + m$, F_1 is such that:

$$|F_1| < 1 + \frac{(\alpha+m)(\beta+m)\xi}{\gamma'''(m+1)} + \frac{(\alpha+m)(\alpha+m+1)(\beta+m)(\beta+m+1)\,\xi^2}{\gamma'''(\gamma'''+1)(m+1)(m+2)} + \ldots$$

If M is the value of this convergent series, we have $|F_1| < M$, and therefore for all γ values so that $\gamma' + m > 0$, F_1 is equally convergent and represents therefore an analytical, univocal and regular function of γ. The part that precedes the term in F_1 is a rational function of γ, with poles of the first order in the points $\gamma = 0, -1, -2, \ldots -m + 1$. Whence, F, considered as a function of γ, is a <u>fractional trascendental</u> function having a pole of the first order in each point $0, -1, -2, \ldots -m, \ldots$, and, consequently, one singular point essential to infinity.

From what is demonstrated in this paragraph, according to the well known/215 principles of the theory of analytical functions it follows that the series F can be differentiated term by term with respect to the parameters α, β, and γ.

4. It is easy to find the various functional properties possessed by the series F due to the special form of its coefficients.

(a) By deriving (1) with respect to x, and by denoting the derivation with respect to this variable by means of accents, we have

$$F' = \frac{\alpha\beta}{\gamma} F(\alpha + 1\ ,\ \beta + 1\ ,\ \gamma + 1\ ,\ x). \qquad (2)$$

(b) we have also

$$xF' = \alpha\left\{\frac{\beta}{\gamma\cdot 1}x + \frac{(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1}x^2 + , \ldots , \right\}$$

in which the terms enclosed by parentheses are

$$F(\alpha+1 , \beta , \gamma , x) - F(\alpha , \beta , \gamma , x).$$

By pointing out the only parameter with which the value can be modified, F is expressed by means of the difference $F(\alpha + 1)-F(\alpha)$, and in the same manner for the other parameters with the expressions:

$$\begin{aligned} xF' &= \alpha\Big(F(\alpha+1) - F(\alpha)\Big), \\ xF' &= \beta\Big(F(\beta+1) - F(\beta)\Big), \\ xF' &= -\frac{1}{\gamma}\Big(F(\gamma+1) - F(\gamma)\Big). \end{aligned} \tag{3}$$

By indicating the finite difference $F(\alpha + 1)-F(\alpha)$ with ΔF and the n^{th} difference with $\Delta^n F$

$$F(\alpha+n) - n F(\alpha+n-1) + \binom{n}{2} F(\alpha+n-2) - \ldots + (-1)^n F(\alpha),$$

from the first equation in (3), by deriving again with respect to x and multiplying by x we obtain:

$$x^2F'' = \alpha(\alpha+1)\Delta^2 F$$

and in general /216

$$x^n F^{(n)} = \alpha(\alpha+1) \ldots \ldots (\alpha+n-1)\Delta^n F. \tag{4}$$

a formula which is easily demonstrated by showing that, supposedly true for the superscript n, it is also valid for the subscript n + 1.

(c) Given the linear differential equation of the second order

$$(x^2 - x)\varphi''(x) + ((\alpha + \beta + 1)x - \gamma)\varphi'(x) - \alpha\beta\varphi(x) = 0\,. \tag{5}$$

if an attempt is made to integrate it in a series by the method of indeterminate coefficients, placing $\phi(x) = \Sigma K_n x^n$, we readily find that

$$\frac{k_{n+1}}{k_n} = \frac{(n + \alpha)(n + \beta)}{(n + \gamma)(n + 1)}\,,$$

and, therefore, with the arbitrary coefficient k_0 being equal to unity, the series $\phi(x)$ coincides with $F(\alpha, \beta, \gamma, x)$. Substituting in this identity

$$(x^2 - x)F'' + ((\alpha + \beta + 1)x - \gamma)F' + \alpha\beta F = 0$$

the values given by (4) for xF' and x^2F'', the following equation is obtained with a simple reduction

$$(\alpha + 1)\,x - 1)\Delta^2 F + ((\alpha + \beta + 1)x - \gamma)\Delta F + \beta x F = 0\,; \tag{6}$$

so that the hypergeometric series satisfies a differential linear equation of the second order with respect to the argument and a linear equation by the differences with respect to each of the parameters.

(d) Eq. (6) can be given another form by substituting its value $F(\alpha + 1) - F(\alpha)$ for the difference ΔF and so on for the second difference. Thus, the linear equation is obtained

$$(\alpha + 1)(x - 1)F(\alpha + 2) - ((\alpha - \beta + 1)x - 2(\alpha + 1) + \gamma)F(\alpha + 1) - (\alpha - \gamma + 1)F(\alpha) = 0,$$

and substituting $\alpha + n$ for α, and replacing $F(\alpha + n)$ with F_n for brevity, the linear equation, which can be said to be <u>recurring of the second order</u> is obtained.

$$(\alpha + n + 1)(x - 1)F_{n+2} - ((\alpha - \beta + n + 1)x - 2(\alpha + n + 1) + \gamma)F_{n+1} - (\alpha + n - \gamma + 1)F_n = 0. \tag{7}$$

By means of this relation one can successively express F_2, F_3, ... by means of F and F_1 by using an equation of the form /217

$$F_n = P_n F + Q_n F_1 ,$$

P_n and Q_n being rational functions of x and of α. Analogous relations are valid for the other parameters, β and γ. We shall state the express case for equation (7) by saying that the F_n values are a recurring system of functions which is linear and of the second order.

(e) Gauss has called contiguous (functiones contiguae) two functions F in which two of the parameters have the same value and the values of the third parameter differ by one unit. Given $F(\alpha, \beta, \gamma, x)$, the system then has 6 contiguous functions and two of these and the primitive F can be associated in 15 different ways. Among these 15 terms there exist some homogeneous linear relations having coefficients of the first degree in x, three out of them arise from paragraph (d) of the present section and the others are readily obtained by the formula (3).

5. It is known that, according to Legendre, the definite integral is called a <u>Eulerian integral of the first species</u> and is denoted by B(p, q)

$$\int_0^1 u^{p-1}(1-u)^{q-1}\, du$$

where p and q are either positive numbers or complex numbers whose real part is positive; it is also known that if we have the recurring relation with respect to p:

$$B(p+1) = \frac{p}{p+q} B(p) ,$$

then

$$B(p+n, q) = \frac{p(p+1)\ldots(p+n-1)}{(p+q)(p+q-1)\ldots(p+q+n-1)} B(p, q).$$

From this can be deduced

$$\int_0^1 u^{\beta+n-1}(1-u)^{\gamma-\beta-1}du = \frac{\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}du,$$

where the real parts of β and $\gamma-\beta$ are assumed to be positive.

Under such circumstances, let us consider the following binomial series:

$$(1-ux)^{-\alpha} = \sum \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{1\cdot2\cdot3\ldots n} u^n x^n;$$

this development is convergent at the same degree if u is real and its value /218 lies between 0 and 1, including the extremes, while $|x| < 1$. Then it is possible to multiply by $u^{\beta-1}(1-u)^{\gamma-\beta-1}du$ and integrate between 0 and 1. Thus, by retaining the relation just established, we obtain:

$$\int_0^1 u^{\beta-1}(1-ux)^{-\alpha}(1-u)^{\gamma-\beta-1} =$$

$$= \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)1\cdot2\cdot3\ldots n} x^n \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}du,$$

therefore

$$F(\alpha, \beta, \gamma, x) = \frac{1}{B(\beta, \gamma-\beta)}\int_0^1 u^{\beta-1}(1-ux)^{-\alpha}(1-u)^{\gamma-\beta-1}du,$$

i.e., by separating the external factor, the hypergeometric series can be put in the form of a definite integral containing the variable x under the sign.

Having reviewed the main properties of the classic hypergeometric series, the individual theories focused on each of these properties will be studied in subsequent chapters and we shall begin with the theory of recurring linear equations to which equations (6) and (7) refer.

CHAPTER II

LINEAR DIFFERENCE EQUATIONS

6. In this Chapter we shall consider recurring linear equations, i.e., relations in which the values of a function $f(n)$ of n appear linearly, for different values n + 1, n + 2, ... of the variable. The relation will be said to be of the order of r if $f(n)$, $f(n + 1)$, ... $f(n + r)$ enter into it.

This relation would then take the following form

$$f(n+r)+a_{1\cdot n}\, f(n+r-1)+a_{2\cdot n}\, f(n+r-2)+\ldots+a_{r\cdot n} f(n)=b_n\,, \tag{1}$$

and it will be called homogeneous or not depending on whether the b_n value in it is respectively zero or different from zero.

If Δ is the known symbol of the finite difference

$$\Delta f=f(n+1)-f(n)\,,$$

by the known formula* /219

$$f(n+h)=f(n)+h\Delta f(n)+\frac{h(h-1)}{1\cdot 2}\Delta^2 f+\ldots+\Delta^h f, \quad (h=1\,,2\,,\ldots r)$$

equation (1) can be transformed into

$$\Delta^r f+a'_{1\cdot n}\,\Delta^{r-1}f+a'_{2\cdot n}\,\Delta^{r-2}f+\ldots a'_{r\cdot n} f=b_n\,,$$

for this reason equation (1) is also adaptable for a linear difference equation of the order of r.

In the present theory, and as long as no other variable besides n plays a part, by constant is understood any quantity which does not change when n varies by integers: in particular, any periodic function of n with a period equal to unity.

*V. p. e. Cesàro, Analisi algebrica (Algebraic analyses), p. 461.

7. Theorem. "A necessary and sufficient condition so that, among the r functions of n $f_1 . f_2 \ldots f_r$, an identical, homogeneous, linear, relation with constant coefficients can exist, is that the following determinant be zero*".

$$D = \begin{vmatrix} f_1(n) & f_2(n) & \ldots f_r(n) \\ f_1(n+1) & f_2(n+1) & \ldots f_r(n+1) \\ \cdot & \cdot & \cdot \\ f_1(n+r-1) & f_2(n+r-1) & \ldots f_r(n+r-1) \end{vmatrix}$$

(a) The condition is necessary. If indeed, c_1, $c_2, \ldots c_r$ are constant, we have

$$c_1 f_1(n) + c_2 f_2(n) + \ldots + c_r f_r(n) = 0 ,$$

by writing those relations which are deduced by changing n into n + 1, n + 2,... n + r - 1 together with this relation, we obtain homogeneous linear equations between c_1, $c_2, \ldots c_r$, which require that D = 0 in order to coexist.

(b) The condition is sufficient. This is true for two functions. In /220 fact, if

$$\begin{vmatrix} f_1(n) & f_2(n) \\ f_1(n+1) & f_2(n+1) \end{vmatrix} = 0 ,$$

then

$$\frac{f_2(n+1)}{f_1(n+1)} = \frac{f_2(n)}{f_1(n)} ,$$

i.e., $f_2(n):f_1(n)$ is a constant in the established sense; if it is placed equal to $c_1:c_2$, we have

*Casorati, Interpreted Calculation of the Finite Differences, etc., Sec. 7. Annali di Matematica, S. II, Vol. 10.

$$c_1 f_1(n) + c_2 f_2(n) = 0.$$

Let us assume now that the proposition is true for r-1 functions. I claim that it is also true for r. Indeed, a very simple identical transformation allows the putting of the determinant D into the form:

$$D = \frac{D_1}{f_1(n+1)\,..f_1(n+r-2)}$$

where D_1 is the determinant

$$\begin{vmatrix} f_2(n)f_1(n+1) - f_1(n)f_2(n+1) & \dots f_r(n)f_1(n+1) - f_1(n)f_r(n+1) \\ f_2(n+1)f_1(n+2) - f_1(n+1)f_2(n+2) & \dots f_r(n+1)f_1(n+2) - f_1(n+1)f_r(n+2) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f_2(n+r-2)f_1(n+r-1)-f_1(n+r-2)f_2(n+r-1) & \dots f_r(n+r-2)f_1(n+r-1)-f_1(n+r-2)f_r(n+r-1) \end{vmatrix}$$

Now here $f_1(n)$ is assumed to be not infinite: therefore D_1 should be identically zero. However, it is a determinant of the same form as D, relative to the r-1 functions

$$f_2(n)\,f_1(n+1) - f_1(n)\,f_2(n+1)\,,\ \dots f_r(n)\,f_1(n+1) - f_1(n)\,f_r(n+1)\,,$$

and therefore an identical, homogeneous linear relation will exist among these:

$$c'_1\left(f_2(n)f_1(n+1)-f_1(n)f_2(n+1)\right) + \dots + c'_{r-1}\left(f_r(n)f_1(n+1)-f_1(n)f_r(n+1)\right) = 0$$

wherefore /221

$$\begin{vmatrix} f_1(n) & c'_1 f_2(n) + \dots + c'_{r-1} f_r(n) \\ f_1(n+1) & c'_1 f_2(n+1) + \dots + c'_{r-1} f_r(n+1) \end{vmatrix} = 0\,;$$

but this being a determinant D relative to the functions

$$f_1(n)\ ,\ c_1'(n)\,f_2(n) + \dots + c'_{r-1} f_r(n)\,,$$

it follows that an identical homogeneous linear relation with constant coefficients, will exist among these and namely, among the $f_1(n)$, $f_2(n)$,... $f_r(n)$, q.e.d.

8. Given an equation of the form (1), each one of its solutions is said to be its integral. Considering from now on the homogeneous equation of order r:

$$f(n+r) + a_{1\cdot n} f(n+r-1) + \ldots + a_{r\cdot n} f(n) = 0\,, \tag{2}$$

in which we shall concede that, at least from an n value on, $a_{r\cdot n}$ is not zero, it is evident that if $\phi(n)$ is one of its integrals, C $\phi(n)$ is also, C being a constant. If $\phi_1(n)$, $\phi_2(n)$ are two integrals of (2), $C_1\phi_1(n) + C_2\phi_2(n)$ will also be an integral, c_1, c_2, also being constants.

Theorem. "Every equation of the form (2) has r linearly independent integrals, i.e., among which no homogeneous linear relation with constant coefficients exists; every other integral of the same equation is associated with the preceding r values by a homogeneous linear relation with constant coefficients".

Let z be a special value of n*: if arbitrary values are given to $f(z)$, $f(z+1)$,... $f(z+r-1)$, Equation (2) allows us to obtain $f(z+r)$, $f(z+r+1)$,..., and with that we shall have an integral of (2). Similar r integrals can be determined among which no homogeneous linear relation with constant coefficients will exist if the arbitrary values

$$\begin{array}{lll} f_1(z) & f_2(z) & \ldots f_r(z) \\ f_1(z+1) & f_2(z+1) & \ldots f_r(z+1) \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ f_1(z+r-1) & f_2(z+r-1) & \ldots f_r(z+r-1) \end{array}$$

*In our theory here the z value can be assumed without restriction to be an integer and also to be null.

are chosen in such a way that their determinant is non-zero (Sec. 7). Once /222 determined, the linearly independent r integrals $f_1(n)$, $f_2(n)$,... $f_r(n)$,

$$C_1 f_1(n) + C_2 f_2(n) + \ldots + C_r f_r(n)$$

will also obviously be an integral of (2). Moreover, every other integral $\phi(n)$ will be linearly bound to those r integrals because by writing eq. (2) for the integrals $f_1(n)$,..., $f_r(n)$, $\phi(n)$ and by eliminating a, $a_{1 \cdot n}$, $a_{r \cdot n}$ among the equations thus written, we obtain

$$\begin{vmatrix} f_1(n) & f_1(n+1) & \ldots & f_1(n+r) \\ f_2(n) & f_2(n+1) & \ldots & f_2(n+r) \\ \cdot & \cdot & \cdot & \cdot \\ f_r(n) & f_r(n+1) & \ldots & f_r(n+r) \\ \varphi(n) & \varphi(n+1) & \ldots & \varphi(n+r) \end{vmatrix} = 0,$$

with the effective result that an identical, homogeneous linear relation exists among $\phi(n)$, $f_1(n)$,... $f_r(n)$ (q.e.d.)

A system of linearly independent r integrals is called a <u>fundamental system</u> of integrals of Eq. (2). Each integral can be expressed in a homogeneous linear function of r, the others constituting a fundamental system.

If $f_1(n)$, $f_2(n)$,..., $f_r(n)$ constitute a fundamental system, the same kind of system will also be constituted by

$$c_{h \cdot 1} f_1(n) + c_{h \cdot 2} f_2(n) + \ldots + c_{h \cdot r} f_r(n), \qquad (h = 1, 2, \ldots r) \tag{3}$$

$c_{h \cdot k}$ being arbitrary constants, provided their determinant is different than zero. It can be said that (3) effects the substitution ($c_{h \cdot k}$) on the primitive fundamental system.

9. By setting the special value z of n equal to zero, it is often convenient to consider the fundamental system $f_1(n)$, $f_2(n)$,... $f_r(n)$ such that

$$f_1(0) = 1 \quad , \quad f_1(1) = 0 \quad , \quad f_1(2) = 0 \, , \ldots \, f_1(r-1) = 0 \, ,$$

$$f_2(0) = 0 \quad , \quad f_2(1) = 1 \quad , \quad f_2(2) = 0 \, , \ldots \, f_2(r-1) = 0 \, ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f_r(0) = 0 \quad , \quad f_r(1) = 0 \quad , \quad f_r(2) = 0 \, , \ldots \, f_r(r-1) = 1 .$$

Such a system will be called "principal". Every other integral $\phi(n)$ /223 can be obviously placed in the form

$$\varphi(n) = \varphi(0) f_1(n) + \varphi(1) f_2(n) + \ldots + \varphi(r-1) f_r(n).$$

The determinant D formed with the principal system has one outstanding property. We set

$$D(n) = \begin{vmatrix} f_1(n) & f_2(n) & \ldots f_r(n) \\ f_1(n+1) & f_2(n+1) & \ldots f_r(n+1) \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ f_1(n+r-1) & f_2(n+r-1) & \ldots f_r(n+r-1) \end{vmatrix};$$

by multiplying respectively by $a_{r-1.n}, a_{r-2.n}, \ldots, a_{1.n}$ the second, third,... and last line and summing with the first multiplied by $a_{r.n}$, we obtain, because of Eq. (2),

$$D(n+1) = (-1)^r a_{r \cdot n} D(n) ,$$

then if $a_{r.n}$ is not zero from n = 0 on, and noting that D(0) = 1, we have

$$D(n) = (-1)^{nr} a_{r \cdot 0} a_{r \cdot 1} \ldots a_{r \cdot n-1} . \tag{4}$$

10. It is known from the elements of the theory of the power series of one variable that these series can give rise to three cases: they are convergent for each finite value of the variable, or for all values of the variable the modulus of which is less than a determined positive number or they are not

convergent for any value of the variable which differs from zero. Now we propose to show that if a succession of numbers k_n is defined for a recurring equation of the form (2), the coefficients $a_{1 \cdot n}$, $a_{2 \cdot n}, \ldots$ $a_{r \cdot n}$ of which have, for each value of n, a modulus smaller than a positive number M, the power series $\Sigma k_n z^n$ definitely does not belong to the third case, i.e., it admits a circle of convergence of finite or infinite radius, but not zero. Indeed, let it be

$$k_{n+r} = a_{1 \cdot n} k_{n+r-1} + a_{2 \cdot n} k_{n+r-2} + \ldots + a_{r \cdot n} k_n.$$

If the value of k_0, $k_1, \ldots$, k_{r-1} are arbitrarily chosen, two positive numbers A and R can always be assigned such that

$$|k_0| < A \ , \ |k_1| < AR^2 \ , \ |k_2| < AR^4 , \ldots , |k_{r-1}| < AR^{2(r-1)} ;$$

moreover being able to choose R larger than M and larger than the only posi- /224 tive root of the equation

$$x^{2r} - x^{2r-2} - x^{2r-4} - \ldots - x^2 - 1 = 0.$$

Then we have

$$|k_r| < AM(1 + R^2 + R^4 + \ldots + R^{2r-2}) < AMR^{2r-1} < AR^{2r} ,$$

then

$$|k_{r+1}| < AM(R^2 + R^4 + \ldots + R^{2r}) < AMR^{2r+1} < AR^{2r+2} ,$$

and thus, in general, $|k_n| < AR^{2n}$. The series $\Sigma k_n z^n$ then converges at least within the circle of radius $1/R^2$, q.e.d.

11. The preceding proposition can be generalized. If the succession of numbers k_n is definite for an inhomogeneous recurring equation

$$k_{n+r} = a_{1 \cdot n} k_{n+r-1} + a_{2 \cdot n} k_{n+r-2} + \ldots + a_{r \cdot n} k_n + b_n ,$$

where the coefficients $a_{1 \cdot n}$, $a_{2 \cdot n}, \ldots$ $a_{r \cdot n}$ have their modulus less than a positive M number for each n value, and the series $\Sigma b_n z^n$ admits a circle of convergence with a radius not equal to zero, in such a way that two positive numbers of B and ρ can be assigned so that

$$|b_n| < B\rho^n,$$

I claim that in this case also the power series $\Sigma k_n z^n$ has a radius of convergence which is definitely not zero. The values of k_0, $k_1 \ldots$, k_{r-1} are again arbitrarily chosen, then two positive numbers of A and R are determined so that that

$$|k_0| < A, \ |k_1| < AR^2, \ |k_2| < AR^4, \ \ldots, \ |k_{r-1}| < AR^{2(r-1)},$$

also taking MA > B, $R^2 > \rho$, finally R larger than M and larger than the only positive root of the equation

$$x^{2r-1} - x^{2r-2} - x^{2r-4} - \ldots - x^2 - 2 = 0.$$

Then we have

$$|k_r| < AM(1 + R^2 + R^4 + \ldots + R^{2r-2}) + B < AM(2 + R^2 + R^4 + \ldots + R^{2r-2})$$

and because of the hypotheses made on R: /225

$$|k_r| < AMR^{2r-1} < AR^{2r}.$$

Analogously, we have

$$|k_{r+1}| < AM(R^2+R^4 + \ldots + R^{2r}) + B\rho < AMR^2(2 + R^2 + R^4 + \ldots + R^{2r-2})$$

then

$$k_{r+1} < AMR^{2r+1} < AR^{2(r+1)};$$

and in the same way it is shown that for each integer value of n, we have $|k_n| < AR^{2n}$: consequently the series $\Sigma k_n z^n$ converges at least within the circle of center z = 0 and of radius $1/R^2$, q.e.d.

12. When r functions of the integer n are given, linearly independent of

each other, $f_1(n)$, $f_2(n), \ldots, f_r(n)$, it is always possible to construct the difference equation of the form (2), for which the given functions constitute a fundamental system. Indeed, every other integral F(n) of that equation will be associated by a homogeneous linear relation with constant coefficients with $f_1(n)$, $f_2(n), \ldots f_r$: it follows then that (Sec. 7) will be

$$\begin{vmatrix} F(n) & f_1(n) & f_2(n) & \ldots f_r(n) \\ F(n+1) & f_1(n+1) & f_2(n+1) & \ldots f_r(n+1) \\ \cdot & \cdot & \cdot & \cdot \\ F(n+r) & f_1(n+r) & f_2(n+r) & \ldots f_r(n+r) \end{vmatrix} = 0,$$

and this is the required equation.

13. Again indicating a fundamental system of integrals of Eq. (2) by $f_1(n)$, $f_2(n), \ldots f_r(n)$, every other one of its integrals can be put in the form

$$c_1 f_1(n) + c_2 f_2(n) + \ldots + c_r f_r(n).$$

Now, as will be evident from the following, it is very important for the case in which the equation has an integral with the property that its ratio to each other integral of the same equation tends to zero when n = ∞. The integral having this property, if it exists, will be unique* by its own definition: this integral will be the so-called distinct integral** and when Eq. (2) has /226 such an integral, it is said that it defines a convergent algorithm.

When the ratios $f_1 : f_r$, $f_2 : f_r, \ldots, f_{r-1} : f_r$ have been determined as well as the finite limits for n = ∞, the quest for the distinct integral of Eq. (2) can be reduced to a similar search for an equation of the same form, but of

*Two integrals in one equation of the type (2) are not considered different if their ratio is a constant.

**On the generation of recurrent systems, etc. Acta Mathematica, Vol. 16, p. 341.

an order which is smaller by one unit.

If we set indeed

$$\lim_{n=\infty} \frac{f_i(n)}{f_r(n)} = \alpha_i$$

and consider the functions of n

$$f_1(n) - \alpha_1 f_r(n) \quad , \quad f_2(n) - \alpha_2 f_r(n) \; , \; \ldots \; , \; f_{r-1}(n) - \alpha_{r-1} f_r(n) \; ;$$

in the light of what was said in Sec. 12, we can construct the difference equation of the order r-1, of which these r-1 functions, clearly without a linear relation, constitute a fundamental system. Now if this equation admits of the distinct integral, it is clear that this would be the distinct integral also for the primitive equation and vice versa.

14. Let us give special consideration to the homogeneous equation of the form (2) in which the coefficients are constants, namely

$$f_{n+r} + a_1 f_{n+r-1} + \ldots + a_r f_n = 0. \tag{5}$$

If α_i is a root of the algebraic equation

$$z^r + a_1 z^{r-1} + a_2 z^{r-2} + \ldots + a_r = 0 \; , \tag{6}$$

it is obvious that α_i^n will be the integral of (5): if, therefore, all the roots of (6) are distinct, the general integral of (5) will have the form

$$f_n = c_1 \alpha_1^n + c_2 \alpha_2^n + \ldots + c_r \alpha_r^n.$$

If there is an α_r among these roots, the modulus of which is less than that of any other, α_r^n gives us the distinct integral.

When (6) does not have all distinct roots, but h of them are equal to α, it is readily evident that the integral containing h arbitrary constants, /227
corresponding to these h roots, is given by

$$\alpha^n(c_1 + c_2 n + c_3 n^2 + \ldots + c_h n^{h-1}). \quad (^*)$$

If the roots of (6) are distinct, a linear relation with constant coefficients cannot exist among the α_1^n, $\alpha_2^n, \ldots \alpha_r^n$.

$$C_1 \alpha_1^n + C_2 \alpha_2^n + \ldots C_r \alpha_r^n = 0.$$

By forming the determinant D for the α_1^n, $\alpha_2^n, \ldots, \alpha_r^n$, the well-known Vandermonde determinant, different than zero, is obtained. If one of the roots of α_1 is multiple of the order of h, a similar relation cannot exist among α_1^n, $n\alpha_1^n$, ..., $n^{h-1}\alpha_1^n$, $\alpha_h^n, \ldots \alpha_r^n$ either, because here also in forming the determinant D, the theory of determinants readily* demonstrates that its value is different than zero. To furnish an example upon which an easy demonstration can be given in the general case, let α_1 be a double root and the others, α_3, $\alpha_4, \ldots \alpha_n$, be simple roots. The determinant D is then

$$\begin{vmatrix} 1 & l & 1 & 1 \\ \alpha_1 & (l+1)\alpha_1 & \alpha_3 & \ldots \alpha_r \\ \alpha_1^2 & (l+2)\alpha_1^2 & \alpha_3^2 & \ldots \alpha_r^2 \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{r-1} & (l+r-1)\alpha_1^{r-1} & \alpha_3^{r-1} & \ldots \alpha_r^{r-1} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & \ldots 1 \\ \alpha_1 & 1 & \alpha_3 & \ldots \alpha_r \\ \alpha_1^2 & 2\alpha_1 & \alpha_3^2 & \ldots \alpha_r^2 \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{r-1} & (r-1)\alpha_1^{r-2} & \alpha_3^{r-1} & \ldots \alpha_r^{r-1} \end{vmatrix}$$

Now on considering the system of r equations with respect to the r un- /228 knowns α_1, $\alpha_3, \ldots \alpha_r$, this system is determined and therefore its determinant, which coincides with D, is different than zero, q.e.d.

---

*Casorati, loc. cit., Sec. 6. The method of decomposition of (5) is used there in symbolic linear factors, quite obvious in the case of constant coefficients. However, this method can also be applied to equations with variable coefficients as I have demonstrated in the two memorandums "On the Difference Equations" R. C. of the Reale Accademia dei Lincei, January 7 and February 4, 1894.

$$\begin{cases} \alpha_1^r + a_1 \alpha_1^{r-1} + \quad \dots + a_r = 0 \\ r\alpha_1^{r-1} + (r-1) a_1 \alpha_1^{r-2} + \dots + a_{r-1} = 0 \\ \alpha_3^r + a_1 \alpha_3^{r-1} + \quad \dots + a_r = 0 \\ \dots\dots\dots\dots\dots\dots \\ \alpha_r^r + a_1 \alpha_r^{r-1} + \quad \dots + a_r = 0 \end{cases}$$

CHAPTER III

SECOND ORDER LINEAR DIFFERENCE EQUATIONS

15. In this chapter we shall apply the facts pertaining to equations of the second order, which we are going to study more thoroughly. We shall write the recurring equation in the form

$$f_{n+2} = a_n f_{n+1} + b_n f_n , \tag{1}$$

and we shall suppose that b_n is not null from an n on, and for example precisely from n = 0. Let us denote the principal system of integrals of this equation by A_n, B_n, i.e., that system for which the initial values are

$$A_0 = 1 \ , \ A_1 = 0 \quad , \quad B_0 = 0 \ , \ B_1 = 1 \ ;$$

every other one of its integrals can be written (Sec. 9)

$$f_n = f_0 A_n + f_1 B_n. \tag{2}$$

The property found for the determinant D(n) by Sec. 9 yields, for the case of the equation of the second order:

$$D(n) = A_n B_{n+1} - B_n A_{n+1} = (-1)^n b_0 b_1 \dots b_{n-1} ,$$

then

$$\frac{A_n}{B_n} - \frac{A_{n+1}}{B_{n+1}} = (-1)^n \frac{b_0 b_1 \dots b_{n-1}}{B_n B_{n+1}}$$

and from this /229

$$\frac{A_n}{B_n} - \frac{A_{n+r}}{B_{n+r}} = (-1)^n b_0 b_1 \ldots b_{n-1} \sum_{\nu=0}^{r-1} (-1)^\nu \frac{b_n b_{n+1} \ldots b_{n+\nu-1}}{B_{n+\nu} B_{n+\nu+1}}. \tag{3}$$

It is now easy to show that:

"The necessary and sufficient condition for the existence of the distinct integral of (1) is that the ratio $A_n:B_n$ has a limit for $n = \infty$."

If λ is this limit, it is immediately evident that the ratio of $A_n - \lambda B_n$ to every other integral $hA_n + kB_n$ of the equation tends to zero when $n = \infty$, and vice versa, if the distinct integral σ_n exists, this can be expressed in a function of A_n and B_n in accordance with (2)

$$\sigma_n = \sigma_0 A_n + \sigma_1 B_n,$$

then, dividing by B_n and passing to the limit for $n = \infty$,

$$\lim \frac{\sigma_n}{B_n} = \lim \left(\sigma_0 \frac{A_n}{B_n} + \sigma_1\right) = 0 \text{ whence } \lim \frac{A_n}{B_n} = -\frac{\sigma_1}{\sigma_0}.$$

However, the condition of the existence of the limit for A_n/B_n is that $A_n/B_n - A_{n+r}/B_{n+r}$ tends to zero for $n = \infty$, i.e., it coincides with the condition of convergence of the series

$$\sigma = \sum_{\nu=2}^{\infty} (-1)^\nu \frac{b_0 b_1 \ldots b_{\nu-2}}{B_{\nu-1} B_\nu}, \tag{3'}$$

the remainder of which is, from (3),

$$\frac{A_n}{B_n} - \sigma = (-1)^n b_0 b_1 \ldots b_{n-1} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{b_n b_{n+1} \ldots b_{n+\nu-1}}{B_{n+\nu} B_{n+\nu+1}};$$

we then have the distinct integral given by

$$\sigma_n = -A_n + \sigma B_n.$$

All this is but the condition of convergence of the continuous fraction /230

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{c_2 + \cdots}}} , \qquad (3'C)$$

in a new form which, compared to the usual form, has the advantage of adapting itself to the extension of recurring equations of any order. The numerators and denominators of (3'C) do not differ from A_2, A_3,... A_n,... and B_2, B_3,... B_n,..., respectively. The reduced (virtual) quotients A_0/B_0 and A_1/B_1 are 1/0 and 0/1, respectively.

The condition of existence of the distinct integral coincides with that of the convergence of the continuous fraction. The value σ now under consideration coincides with the value of the continuous fraction and the distinct integral σ_n then is defined by the initial condition $\sigma_1 : \sigma_o = -\sigma$.

16. Poincaré has been credited with a proposition which has great usefulness in the applications of the theory of recurring equations. We shall demonstrate this proposition for the case of equations of the second order, referring the reader to the original memorandum* for the demonstration in the case of equations of any order.

Poincaré's Theorem. "Let $\lim_{n=\infty} a_n = a$, and $\lim_{n=\infty} b_n = b$ in Equation (1). The equation

$$t^2 - at - b = 0 , \qquad (4)$$

which is said to be characteristic of (1) has roots α and β and $|\alpha| > |\beta|$. The limit of the ratio $f_{n+1} : f_n$ for an integral f_n of (1) is generally equal to α and exceptionally equal to β."

*On Linear Equations, etc. American Journal of Mathematics, Vol. 7, No. 3, 1885.

(a) We put, f_n being an integral of (1):

$$f_n = X_n + Y_n \quad , \quad f_{n+1} = \alpha X_n + \beta Y_n ,$$

from which we deduce

$$f_{n+1} = X_{n+1} + Y_{n+1} \quad , \quad f_{n+2} = \alpha X_{n+1} + \beta Y_{n+1} .$$

From these we derive, taking into account Equation (1): /231

$$X_{n+1} = \frac{(a_n \alpha + b_n - \beta\alpha) X_n + (a_n \beta + b_n - \beta^2) Y_n}{\alpha - \beta} ,$$

$$Y_{n+1} = \frac{(\alpha^2 - a_n \alpha - b_n) X_n + (\alpha\beta - a_n \beta - b_n) Y_n}{\alpha - \beta}$$

and putting

$$\frac{\alpha^2 - a_n \alpha - b_n}{\alpha - \beta} = A_n \quad , \quad \frac{\beta^2 - a_n \beta - b_n}{\alpha - \beta} = B_n ,$$

we obtain

$$\begin{cases} X_{n+1} = \alpha X_n - (A_n X_n + B_n Y_n) , \\ Y_{n+1} = \beta Y_n + (A_n X_n + B_n Y_n) , \end{cases}$$

where A_n, B_n tend to zero when $n = \infty$. Having placed $G_n = Y_n : X_n$, we have, on dividing the preceding equations term by term:

$$G_{n+1} = G_n \frac{\beta + B_n + A_n \frac{1}{G_n}}{\alpha - A_n - B_n G_n} . \tag{5}$$

(b) A positive number λ can now be determined, which tends to zero with increasing n and such that for each positive number k between λ and $1/\lambda$

$$\left|\beta + B_n + A_n \frac{1}{k}\right| < \left|\alpha - A_n - B_n k\right|,$$

from one n onward. The preceding inequality is certainly satisfied if

$$|\beta| + \varepsilon\left(1 + \frac{1}{k}\right) < |\alpha| - \varepsilon(1 + k),$$

ε being a positive number superior to $|A_n|$, $|B_n|$ and that it can be assumed to be arbitrarily small for a sufficiently large n: it is sufficient to require

$$k < \frac{|\alpha| - |\beta|}{2\varepsilon} - 1 \quad , \quad \frac{1}{k} < \frac{|\alpha| - |\beta|}{2\varepsilon} - 1$$

so that the required λ number is /232

$$\lambda = \frac{2\varepsilon}{|\alpha| - |\beta| - 2\varepsilon}.$$

(c) Let ε be taken arbitrarily small and n sufficiently large in order to have:

$$|A_n| < \varepsilon \ , \ |B_n| < \varepsilon,$$

if $|G_n|$ is also less than ε, we shall have from (5)

$$|G_{n+1}| < \frac{(|\beta| + 1 + \varepsilon)\varepsilon}{|\alpha| - \varepsilon(1 + \varepsilon)},$$

and this can be made arbitrarily small, and in any event less than unity.

(d) This having been established, let us examine in what manner G_n is varying with increasing n. Three hypotheses can be made:

<u>1st</u>. For a given n, $|G_n|$ is less than $\lambda < \varepsilon$. Then, through (c), $|G_{n+1}| < 1$, therefore, it will also be less than λ or it will be between λ and $1/\lambda$.

2nd. Let $|G_n|$ be between λ and $1/\lambda$. It then results from (b) that

$$\left|\beta + B_n + A_n \frac{1}{G_n}\right| < |\alpha - A_n - B_n G_n|$$

and therefore $|G_{n+1}/G_n|$ is less than a quantity less than one and it remains so as n increases.

3rd. Let finally $|G_n|$ be greater than $1/\lambda$. Then $|G_{n+1}|$ either remains so also or it is less than $1/\lambda$, reentering into one of the two preceding hypotheses.

It is evident from this analysis that if $|G_n|$ becomes less than $1/\lambda$ for one n value, it would tend to zero when $n = \infty$. Except for this case, it is possible that $|G_n|$ may be always greater than $1/\lambda$, and then its limit is infinite when $n = \infty$.

(e) However we have

$$\frac{f_{n+1}}{f_n} = \frac{\alpha X_n + \beta Y_n}{X_n + Y_n}.$$

in the event that $G_n = Y_n : X_n$ tends to zero when $n = \infty$, we obtain from this: /233

$$\lim_{n=\infty} \frac{f_{n+1}}{f_n} = \alpha;$$

in the event that G_n tends to infinity, we then have

$$\lim_{n=\infty} \frac{f_{n+1}}{f_n} = \beta,$$

q.e.d.

17. Henceforth, we shall allow Equation (1) to contain linearly a variable x in its coefficient a_n,

$$\alpha_n = a_n' x + a_n'',$$

so that (1) will be written:

$$f_{n+2} = (a_n' x + a_n'') f_{n+1} + b_n f_n. \tag{5}$$

The integrals A_n and B_n in this case are whole number rational polynomials in x and it is readily evident that A_n is of the rank of n - 2 and B_n of the rank n - 1.

It is evident from Sec. 15 that the necessary and sufficient condition for the existence of the distinct integral of (5) is that the following series be convergent

$$\sigma = \sum_{\nu=2}^{\infty} (-1)^{\nu} \frac{b_0 b_1 \ldots b_{\nu-2}}{B_{\nu-1} B_{\nu}};$$

having made this hypothesis, we have

$$\frac{A_n}{B_n} - \sigma = (-1)^n \left(\frac{b_0 b_1 \ldots b_{n-1}}{B_n B_{n+1}} - \frac{b_0 b_1 \ldots b_n}{B_{n+1} B_{n+2}} + \ldots \right).$$

If now the series involved in this second member and the terms of which are functions of x, it is expanded - formally - in a negative power series of x, it is immediately evident that this series takes the form

$$\frac{c}{x^{2n-1}} + \frac{c'}{x^{2n}} + \frac{c''}{x^{2n+1}} + \ldots,$$

since B_n is of the rank n - 1 in x. We shall express this fact by saying that such a series is of the rank -2n+1 in x. Upon forming then the distinct integral $\sigma_n = A_n - \sigma B_n$, this integral will be of the rank -n. Consequently: /234

"The distinct integral of (5) is formally representable by means of a system of negative power series of x, of the respective rank -n."

It can easily be demonstrated that Equation (5) cannot have a second integral of this form since the condition that $f_1, f_2, \ldots f_n, \ldots$ be a negative power series of x of the ranks -1, -2,... -n,... is sufficient (if $f_0 = 1$) to determine unambiguously and successively all the coefficients in this series.

Now it remains to be seen if the developments in negative power series, obtained formally in this manner for the σ_n values, have another significant effect: this will be evident from the subsequent Sections, in which it will be demonstrated that "if a_n', a_n'', b_n have finite limits when $n = \infty$, a circle with a center in the origin and with a finite radius can always be assigned in the plane of the complex variable x. Outside of this circle the distinct integral through (5) not only does exist, but this integral is also constituted by a system σ_n of the negative power series of x, of the rank -n and convergent outside of the said circle".

18. (a) With this effect we shall assume that none of the a'_n values nor their limits can be null: then a simple change of variables permits of unrestrictedly setting

$$\lim_{n=\infty} a_n' = 2 \quad , \quad \lim_{n=\infty} a_n'' = 0 \quad , \quad \lim_{n=\infty} b_n = 1 .$$

The characteristic equation (4) of (5) thus becomes

$$z^2 - 2xz + 1 = 0 , \tag{6}$$

having two roots which will be denoted by $\alpha(x)$ and $1/\alpha(x)$; the modulus of $\alpha(x)$ being ρ, $\rho > 1$ can be assumed for each x value, with the exception of the real x values and those between -1 and +1. The x values by which ρ is maintained constant are found on an ellipse with foci at the points +1 and -1.

(b) All the a_n' values being different than zero, their moduli will have a lower limit which can be 2 and in a different case will be an effective minimum, different from zero which we shall denote by A'. The moduli of a_n'' and b_n will have superior limits which we shall indicate by A'' and B. Finally let η be an arbitrarily small, but fixed, positive quantity.

We describe a circle of center x = 0 and of radius

$$R = \frac{A'' + B + \eta + 1}{A'},$$

in the plane of the complex variable x and I shall call this circle R. Being /235 that $B_1 = 1$; $B_2 = (a_0'x + a_0'')B_1$, if $|x| > R$, we shall have

$$\left|\frac{B_2}{B}\right| > B + \eta + 1 > 1 + \eta;$$

thus

$$\left|\frac{B_3}{B_2}\right| = \left|a_1'x + a_1'' + b_1\frac{B_1}{B_2}\right| \geq |a_1'x| - |a_1''| - \left|b_1\frac{B_1}{B_2}\right| > 1 + \eta.$$

If we assume demonstrated up to a given index the inequality

$$\left|\frac{B_{n+1}}{B_n}\right| > 1 + \eta,$$

this is true for the following index, always assuming that $|x| > R$, since

$$\left|\frac{B_{n+2}}{B_{n+1}}\right| = \left|a_n'x + a_n'' + b_n\frac{B_n}{B_{n+1}}\right| \geq |a_n'x| - |a_n''| - \left|b_n\frac{B_n}{B_{n+1}}\right| > 1 + \eta.$$

However, Poincaré's theorem teaches us that the ratio $B_{n+1} \cdot B_n$ approaches one of the roots of Equation (6): therefore, the limit of this ratio can only be the root α of the modulus greater than unity; we conclude, namely, that with the values $|x| > R$, we have

$$\lim_{n=\infty} \frac{B_{n+1}}{B_n} = \alpha(x).$$

(c) Finally let us note that for the values $|x| > R$, the $B_n(x)$ cannot have any zero values. If indeed we have $B_{n+2} = 0$ for such a value of x, the following equation would result

$$a_n'x + a_n'' = -\frac{b_n B_n}{B_{n+1}},$$

an impossible result because the modulus of the first member is greater than $B + \eta + 1$, while that of the second member is less than B.

19. The considerations developed in the preceding Section now permit us to demonstrate that

"For values of x outside of circle R the continuous fraction defined by Equation (5) is convergent (in other words Equation (5) has a 'distinct integral')."

We note therefore that the value σ of the continuous fraction is given by series (3') of which it then suffices to show the convergence. If we consider now the terms of the above mentioned series by their absolute values, the ratio between one term and the preceding one is given by /236

$$\frac{b_{\nu-1} B_{\nu-1}}{B_{\nu+1}}$$

and this tends to the limit $1/|\alpha^2(x)| = 1/\rho^2$ when $\nu \to \infty$, and where $\rho > 1$.

Having then excluded the possibility that the terms of the series can be infinite (Sec. 18, c) for values of x outside of circle R, we conclude that for such values of x the series (3') converges absolutely, q.e.d.

20. The series (3') can be again shown to be convergent to the same degree outside of circle R.

In fact, the modulus of the ratio of one term of this series to the preceding one is

$$b_\nu \frac{B_{\nu-1}}{B_\nu} < \frac{B}{(1+\eta)^2};$$

now the quantity η, which can be chosen arbitrarily, provided R is selected properly (Sec. 18, b), can be made to be (k being a positive number less than unity)

$$\eta > \sqrt{\frac{B}{k}} - 1$$

then it follows that

$$\left| b_\nu \frac{B_{\nu-1}}{B_\nu} \right| < k < 1.$$

Since R has been chosen in accordance with the value thus fixed for η, and k is independent of x, the result is that (3') converges to the same degree outside of circle R. The series denoted by Sec. 17 with σ_n has the same property which is composed by the remainders of σ.

However, since the series σ and σ_n are convergent to the same degree when $|x| > R$, a known theorem on the theory of functions* shows that these series are consequently analytic and regular functions of x in that region and, as such, they can be expanded in series of decreasing powers of x. The proposition

*Weierstrass: The Theory of Functions (Zur Functionenlehre). Monatsberichte der Akad. der Wissensch. zu Berlin, 1881.

that we have enunciated at the conclusion of Sec. 17 thus remains established, namely that "the distinct integral of (5) not only exists outside of circle R, but coincides with the unique integral represented by the negative power /237 series σ_n of x of rank -n, which converges outside of the same circle*".

21. The propositions demonstrated in the preceding Sections now permit applying to $A_n(x)$, $B_n(x)$ and $\sigma_n(x)$ all the properties that there are for the numerators, denominators and remainders of the reductions appearing in the expansion of a given function in an algebraic continuous fraction. We shall not discuss these properties further because the reader can secure information on them by consulting the first part of Vol. II of the Handbook of Spherical Functions by Heine as well as Chapter V of the first part of Vol. I, and the book by Possé: On Some Applications of Algebraic Continuous Fractions (St. Petersburg, 1886). We shall limit ourselves here to formally establishing an expansion which we will have to refer to in the last chapter of this work, where the convergence conditions for a case with considerable generality will be given. It is noteworthy that, as far as I know, nothing has been said in general about the convergence conditions of such an expansion. Let us rewrite Equation (5) in the form

$$a_n f_{n+2} + (b_n' x + b_n'') f_{n+1} + c_n f_n = 0 , \tag{5'}$$

and together with this let us consider the other equation, which we shall call its inverse

*The studies related in this chapter, as those of the following chapter also, can be extended with the same methods to recurring equations of order greater than the second. In the recent Memorandum: A Contribution to the Generalization of Continuous Fractions (Memo. of the Academy of Sciences of Bologna, S. V., Vol. 4, 1894), I have also demonstrated the existence of the distinct integral and its representation by means of a negative power series of x, for sufficiently large values of $|x|$ for equations of the third order in a way analogous to (5).

$$a_{n-1} f_n + (b_n' z + b_n'') f_{n+1} + c_{n-1} f_{n+2} = 0. \tag{5''}$$

Let $S_n(z)$ be an integral of the latter equation, for which the initial condition

$$c = (b_0' z + b_0'') S_1 + c_1 S_2.$$

holds. Then by multiplying (5') by $S_{n+1}(z)$ and by summing over all the values of n from zero to infinity, and by considering for f_n the integral $B_n(x)$ determined by $B_0 = 0$, $B_1 = 1$, we have:

$$-x \sum_{n=1}^{\infty} b'_{n-1} B_n(x) S_n(z) = B_1(b_0'' S_1 + c_1 S_2) + B_2(a_0 S_1 + b_1'' S_2 + c_2 S_3) + \dots$$

now, upon taking (5") and the initial condition stated above into account, /238
we obtain

$$-x \sum_{n=1}^{\infty} b'_{n-1} B_n(x) S_n(z) = c - b_0' z S_1 - b_1' z S_2 B_2 - \dots$$

or

$$(z - x) \sum_{n=1}^{\infty} b'_{n-1} B_n(x) S_n(z) = c$$

hence

$$\frac{1}{z - x} = \frac{1}{c} \sum_{n=1}^{\infty} b'_{n-1} B_n(x) S_n(z). \tag{7}$$

This is the formal expansion that we wished to establish* and the effective validity of which will be demonstrated in Sec. 55 for the case in which a_n, b_n', b_n'', c are whole integer rational functions of the subscript n.

---

*Heine gives this development in the case of $a_n = c_n = 1$ (op. cit., Vol. I, p. 203) but he limited his comments to: <u>the convergence is still assumed</u>. It is also presented by Jordan, Cours d'Analyse, 2nd edition, Vol. II, p. 259. In the works cited equation (7) was established by another method: the one used by us has the advantage of being able to be extended readily to include recurring equations of any order.

CHAPTER IV

LINEAR DIFFERENTIAL EQUATIONS*

22. Theorem. "A necessary and sufficient condition so that an identical, homogeneous and linear relation with constant coefficients exists between r analytic functions of x, ϕ_1, ϕ_2,... ϕ_r, which are regular in a common interval of values of the variable, is that the determinant /239

$$D(x)=\begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_r \\ \varphi_1' & \varphi_2' & \dots & \varphi_r' \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_1^{(r-1)} & \varphi_2^{(r-1)} & \dots & \varphi_r^{(r-1)} \end{vmatrix}.$$

is zero, in which the derivatives with respect to x are denoted by superscripts".

(a) The condition is necessary. If indeed, c_1, c_2,... c_r being constants, we have

$$c_1\varphi_1 + c_2\varphi_2 + \dots c_r\varphi_r = 0 \,,$$

it is sufficient to derive r - 1 times with respect to x and eliminate c_1, c_2, ... c_r to see that D(x) = 0.

(b) The condition is sufficient. Let D = 0, the reciprocals of the last line being ≠0, otherwise the theorem would be demonstrated by one of them. Upon deriving D with the rule for the derivation of determinants, all the determinants which are obtained are identically zero except for one, so that we have

*In this chapter we are speaking exclusively of analytic functions with the variable x, although some of the propositions found here, especially those of the first Sections, are also applicable to non-analytic functions, provided that they have derivatives of the first order. The theorems in this chapter originate with Fuchs (Crelle, Vol. 56); the considerations of Sec. 27 stem from Casorati (Memorandum cited).

$$D'(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_r \\ \varphi_1' & \varphi_2' & \dots & \varphi_r' \\ \dots & \dots & \dots & \dots \\ \varphi_1^{(r-2)} & \varphi_2^{(r-2)} & \dots & \varphi_r^{(r-2)} \\ \varphi_1^{(r)} & \varphi_2^{(r)} & \dots & \varphi_r^{(r)} \end{vmatrix}$$

and in order for D to be identically zero, $D'(x) = 0$. Now if $U_1, U_2, \dots U_r$ are the reciprocals of the first line in D: those of the first line in D' will be the respective derivatives and we shall have

$$\begin{cases} U_1\varphi_1 + U_2\varphi_2 + \dots + U_r\varphi_r = 0 \\ \dots\dots\dots\dots\dots \\ U_1\varphi_1^{(r-2)} + U_2\varphi_2^{(r-2)} + \dots + U_r\varphi_r^{(r-2)} = 0 \end{cases}$$

and /240

$$\begin{cases} U_1'\varphi_1 + U_2'\varphi_2 + \dots + U_r'\varphi_r = 0 \\ \dots\dots\dots\dots\dots \\ U_1'\varphi_1^{(r-2)} + U_2'\varphi_2^{(r-2)} + \dots + U_r'\varphi_r^{(r-2)} = 0, \end{cases}$$

then, because not all the determinants of the matrix of this system are zero, we have

$$U_1 : U_2 : \dots : U_r = U_1' : U_2' : \dots : U_r'$$

from which

$$\frac{d \log U_h}{dx} = \frac{d \log U_k}{dx}, \quad U_h : U_k = \text{constant},$$

and therefore a linear relation with constant coefficients exists between ϕ_1, $\phi_2, \dots \phi_r$, q.e.d.

23. We shall consider homogeneous linear differential equations of the

n^{th} order in the following:

$$A_0 y^{(n)} + A_1 y^{(n-1)} + \ldots A_n y = 0 , \tag{1}$$

where A_0, $A_1 \ldots A_n$ are analytic functions of x, regular in a common interval of values of the variable.

(a) An equation of this type has as many linearly independent integrals as the number of units of its order. Assuming this to be true for the equation of order n - 1, let us put

$$y = u\varphi ;$$

substituting in (1)

$$A_0\varphi u^{(n)} + (A_1\varphi + n A_0\varphi') u^{(n-1)} + \ldots = 0 , \tag{2}$$

where the term in u is zero and therefore (2) is a linear differential equation of the order (n - 1) in u' and has, as supposed, (n - 1) linearly independent integrals $\psi_2, \psi_3, \ldots \psi_n$. Let us now consider the functions

$$\varphi , \varphi \int \psi_2 \, dx \ , \ \varphi \int \psi_3 \, dx \ , \ldots , \varphi \int \psi_n \, dx ;$$

all these will be integrals of (1): moreover, if the determinant D(x) is formed for these functions, it is immediately evident that it is equal to /241

$$\varphi^n \begin{vmatrix} \psi_2 & \psi_3 & \ldots \psi_n \\ \psi_2' & \psi_3' & \ldots \psi_n' \\ \cdot & \cdot & \cdot \\ \psi_2^{(n-2)} & \psi_3^{(n-2)} & \ldots \psi_n^{(n-2)} \end{vmatrix} ,$$

for hypotheses different than zero.

(b) If $\phi_1, \phi_2, \ldots,$ are integrals of (1), each expression $c_1\phi_1 + c_2\phi_2 + \ldots$ will also be an integral.

(c) If $\phi_1, \phi_2, \ldots \phi_n$ are n linearly independent integrals of (1), every other integral ϕ will be of the form

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + \ldots c_n\varphi_n :$$

in fact, by substituting the values ϕ, ϕ_1, $\varphi_2, \ldots, \phi_n$ in (1) and eliminating the coefficients A_0, $A_1, \ldots A_n$ among the identities thus written, we obtain

$$\begin{vmatrix} \varphi^{(n)} & \varphi^{(n-1)} & \ldots & \varphi' & \varphi \\ \varphi_1^{(n)} & \varphi_1^{(n-1)} & \ldots & \varphi'_1 & \varphi_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi_n^{(n)} & \varphi_n^{(n-1)} & \ldots & \varphi'_n & \varphi_n \end{vmatrix} = 0$$

with the result (Sec. 22) that a homogeneous linear relation with constant coefficients exists among ϕ, $\varphi_1, \ldots \phi_n$.

24. A system of n integrals of (1), among which no homogeneous linear relation with constant coefficients exists, is called a <u>fundamental</u> <u>system</u> of Equation (1). Let φ_1, $\phi_2, \ldots, \phi_n$ be such a system: if we set

$$\Phi_i = a_{i\cdot 1}\varphi_1 + a_{i\cdot 2}\varphi_2 + \ldots a_{i\cdot n}\varphi_n \quad , \quad (i = 1, 2, \ldots n) \tag{3}$$

the determinant being $\delta = \Sigma \pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{n\cdot n}$, different than zero, the system ϕ_1, $\phi_2, \ldots \phi_n$ will also be a fundamental system and vice versa, each fundamental system is expressed by means of one of these in the form of (3). Let Δ be the determinant formed, as D, with the integrals ϕ_1, $\phi_2, \ldots \phi_n$, it follows from (3) that /242

$$\Delta = \delta D ,$$

which is expressed by saying that D is an <u>invariant</u> of Equation (1). If in the determinant

$$D = \begin{vmatrix} \varphi_1 & \varphi_2 & \dots \varphi_n \\ \varphi'_1 & \varphi'_2 & \dots \varphi'_n \\ \dots & \dots & \dots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \dots \varphi_n^{(n-1)} \end{vmatrix}$$

the last line is multiplied by A_1 and is added to the first, second, ... $n - 1$, multiplied respectively by A_n, A_{n-1},... A_2, we obtain $-A_0 D'(x)$, whence we have for D the notable expression:

$$D(x) = ce^{-\int \frac{A_1}{A_0} dx}. \tag{4}$$

25. Fuchs theorem. "Equation (1) has a fundamental system of integrals which are regular analytic functions in the neighborhood of the point x_0 taken in region T in which the analytic functions

$$\frac{A_1}{A_0}, \frac{A_2}{A_0}, \dots \frac{A_n}{A_0}$$

are regular."

For the sake of brevity we set $A_i/A_0 = B_i$, $(i = 1, 2, \dots n)$ and Equation (1) will be written

$$y^{(n)} + B_1 y^{(n-1)} + B_2 y^{(n-2)} + \dots + B_n y = 0. \tag{1'}$$

The expansion of the functions B_i in the neighborhood of x_0 will now be considered and for simplicity we set $x = 0$ instead of $x = x_0$. By denoting with ρ the radius of a circle of center $s = 0$ within which B_1, B_2,... B_n are regular, we have, when $|x| < \rho$,:

$$B_i = a_{i0} + a_{i1}x + a_{i2} + \dots, \quad (i = 1, 2, \dots, n).$$

We now attempt to satisfy (1') by putting /243

$$y = f(x) = k_0 + k_1 x + k_2 x^2 + \ldots k_\nu x^\nu + \ldots \tag{5}$$

The coefficients $k_0, k_1, \ldots, k_{n-1}$ having been arbitrarily chosen, and noting that $k_\nu = 1/\nu! f^{(\nu)}(0)$, we can determine from (1'), by means of successive derivations, $y^{(n)}, y^{(n+1)}, y^{(n+2)}, \ldots$ as a function of $y, y', \ldots y^{n-1}$, namely k_n, $k_{n+1}, k_{n+2}, \ldots$ as a function of $k_0, k_1, \ldots k_{n-1}$. In the expressions of k_ν as a function of $k_0, k_1, \ldots k_{n-1}$ only operations of summation and multiplication on the quantities $a_{i.\nu}$ and $k_0, k_1, \ldots k_{n-1}$ are involved, these latter entering linearly.

The equation can then be satisfied <u>formally</u> with a expansion of the form (5), containing linearly the n constants $k_1, k_2, \ldots k_\nu$. It remains to be demonstrated that this expansion is convergent in the neighborhood of $x = 0$.

For this purpose let us denote by M a positive number greater than the maximum modulus of $B_1, B_2, \ldots B_n$ within the circle of center $x = 0$ and radius $r < \rho$: by means of a well-known power series theorem we shall have

$$|a_{i \cdot r}| < \frac{M}{r^\nu}.$$

Let us now consider the equation

$$Y^{(n)} = \frac{M}{1 - \frac{x}{r}} (Y^{(n-1)} + Y^{(n-2)} + \ldots + Y' + Y) \tag{6}$$

upon developing in series the coefficient of $Y^{(i)}$, the general term of this series will be $M/r^\nu x^\nu$, therefore greater in absolute value than the corresponding term in B_i. Let us now set

$$Y = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \ldots \lambda_\nu x^\nu + \ldots \tag{7}$$

with $\lambda_0 = |k_0|$, $\lambda_1 = |k_1|, \ldots, \lambda_{n-1} = |k_{n-1}|$; equation (6) will allow us to derive the values of $\lambda_n, \lambda_{n+1}, \ldots$ as a function of $\lambda_0, \lambda_1, \ldots \lambda_{n-1}$: the calculations required are the same as those used to obtain $k_n, k_{n+1}, \ldots$ as a function of k_0, k_1,... k_{n-1} from (1'): only that in the λ_ν, instead of the k_0, k_1,... k_{n-1} values, their moduli are involved, as well as, instead of the $a_{i.\nu}$ the positive quantities M/r^ν, greater in modulus than the corresponding $a_{i.\nu}$, will be involved. Therefore the coefficients λ_n, λ_{n+1},... will all be positive /244 and we shall have $\lambda_\nu > |k_\nu|$. Therefore, in order to prove the convergence of the expansion (5) it is sufficient to prove the one of the expansion (7).

For this purpose we observe that the recurring equation in λ_ν, is obtained from (6) by substituting the expansion (7):

$$(\nu+n)(\nu+n-1)\ldots(\nu+1)\lambda_{\nu+n} = (\nu+n-1)(\nu+n-2)\ldots(\nu+1)\left(\frac{\nu}{r}+M\right)\lambda_{\nu+n-1} +$$

$$+ (\nu+n-2)\ldots(\nu+1)M\lambda_{\nu+n-2} + \ldots (\nu+1)M\lambda_{\nu+1} + M\lambda_\nu,$$

now it is immediately evident from the theorem of Sec. 10 that this equation defines such a system λ_ν that the series $\Sigma\lambda_\nu x^\nu$ has a determined radius (non zero) of convergence: at this point the theorem is proved.

26. Given a linear differential equation of the form (1), a point x_0, in which the functions A_1/A_0, A_2/A_0,... A_n/A_0 are regular, will be called a <u>non-singular</u> point for the equation: every other point will be a <u>singular</u> point. Then we can state, as a consequence of the preceding theorem, that:

"In the neighborhood of each non-singular point of a linear differential equation all of its integrals are regular and analytic functions."

From this, and calling on the general principles of the theory of analytic functions, the analytic continuation of each integral of (1) can be constructed in each connected area in which A_1/A_0, A_2/A_0,... A_n/A_0 do not have singular points, and the analytic continuation of an integral will never cease to be an integral of the same equation.

27. Let x_0 be an isolated singular point of equation (1). In the event

that x_0 is the point to infinity of the sphere-plane of the variable x, we carry out the transformation $x = 1/z$. It is always possible to assign a neighborhood (r) of x_0 within which no other singular point of (1) is found: assuming the counterclockwise rotations to be positive, let the variable x, without leaving the neighborhood (r), execute a positive turn around x_0: when, after execution of this turn, the variable returns to the point of departure x, an integral $\phi(x)$ will have changed in value, generally, and the new value is indicated by $\Phi(x)$. If now it is noted that the function

$$t = \frac{1}{2\pi i} \log(x - x_0) \tag{8}$$

is increased by 1 for such a turn, $\phi(x)$ can be denoted by ϕ_t, $\bar{\phi}(x)$ by ϕ_{t+1}, /245
and thus by φ_{t+2}, φ_{t+3},...which becomes $\phi(x)$ after two, three,...turns of the variable in the positive direction.

Now, these n + 1 functions ϕ_t, ϕ_{t+1},... ϕ_{t+n}, are all integrals of (1) and therefore an homogeneous and linear relation exists between them, whose coefficients are constant with respect to <u>t</u>, namely, functions of x with a single value <u>in the neighborhood of</u> (r); this relation has the form:

$$A_0\varphi_{t+n} + A_1\varphi_{t+n-1} + A_2\varphi_{t+n-2} + \dots + A_n\varphi_t = 0, \tag{9}$$

and with respect to <u>t</u> considered as a variable, it is none other than a <u>homogeneous linear equation with constant coefficients</u>, and as such can be integrated as indicated in Sec. 14. We shall suppose that in this equation ϕ signifies the general integral of (1), with <u>n</u> arbitrary constants.

If, having formed the equation

$$A_0\omega^n + A_1\omega^{n-1} + \dots A_{n-1} + A_n = 0 \tag{10}$$

this has all its roots distinct, ω_1, ω_2,... ω_n, the integral of equation (9) will be given, in accordance with Sec. 14, by

$$u_1 \omega_1^t + u_2 \omega_2^t + \ldots + u_n \omega_n^t. \tag{11}$$

where $u_1, u_2, \ldots u_n$ are constants with respect to t, namely, functions of x with a unique value in the neighborhood of x_0. If equation (9) has multiple roots, and ω is, e.g., a root of multiplicity h, for the corresponding h terms of (10) (Sec. 14) the following will be substituted

$$\omega_1^t (v_1 + v_2 t + \ldots v_h t^{h-1}). \tag{12}$$

Having performed this, the value (8) is substituted in the expressions (11) and (12) for t, and these are changed respectively, putting $\omega_k = e^{2\pi i \rho k}$, into

$$u_1 (x - x_0)^{\rho_1} + u_2 (x - x_0)^{\rho_2} + \ldots + u_n (x - x_0)^{\rho_n} \tag{13}$$

and

$$(x - x_0)^{\rho_1} (v_1 + v'_2 \log (x - x_0) + \ldots v'_h \log^{h-1} (x - x_0)). \tag{14}$$

Now if we substitute y in the first member of the equation (1) by an expres- /246 sion of the form (13), we evidently obtain an expression of the same form, namely, of the form (11) in t. However, this (Sec. 14) cannot be zero unless all its coefficients are zero and the same thing is true if an expression of the form (14) is put in (1), whence it can be concluded that:

"If the equation (10) has all its roots distinct, equation (1) admits n integrals in the neighborhood of x_0, called canonic integrals, which form a fundamental system

$$u_k (x - x_0)^{\rho_k} \quad , \quad \rho_k = \frac{1}{2\pi i} \log \omega_h \quad , \quad (k = 1, 2, \ldots n) \tag{15}$$

where u_k is a function of x with a unique value in the neighborhood (r) of x_0; if equation (10) has a multiple root ω_1, of multiplicity h, (1) has h integrals in the neighborhood of x_0

$$\begin{cases} v_1 (x - x_0)^{\rho_1}, \\ v_1' (x - x_0)^{\rho_1} \log(x - x_0) + v'_2 (x - x_0)^{\rho_1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (v_1^{(h)} \log^h (x-x_0) + v_2^{(h)} \log^{h-1} (x-x_0) + \ldots + v_h^{(h)}) (x-x_0)^{\rho_1}. \end{cases} \tag{16}$$

28. (a) The demonstration of this theorem of Fuchs that was given in the preceding Sec. stems, as stated, from Casorati. Equation (10) has been called the <u>fundamental</u> equation relative to the singular point x_0, and the <u>determinant</u> the equation which has $\rho_1, \rho_2, \ldots \rho_n$ for roots and is deduced from the fundamental by setting $= e^{2\pi i \rho}$.

(b) The following method is suitable for forming the fundamental equation. Let $\phi_1, \phi_2, \ldots \phi_n$ be any fundamental system of (1) and ψ a canonic integral. When x rotates around x_0, ϕ_k will change into a new integral $\bar{\phi}_k$ of (1) and thus

$$\bar{\varphi}_k = \alpha_{k \cdot 1} \varphi_1 + \alpha_{k \cdot 2} \varphi_2 + \ldots + \alpha_{k \cdot n} \varphi_n \quad (k = 1, 2, \ldots n); \tag{17}$$

these formulas give the <u>linear substitution</u> which the system $(\phi_1, \phi_2, \ldots \phi_n)$ undergoes after one rotation of the variable around x_0. Now having

$$\bar{\psi} = \omega \psi,$$

it will be, having set /247

$$\psi = c_1 \varphi_1 + c_2 \varphi_2 + \ldots c_n \varphi_n.$$

substituting (17) and noting that a homogeneous linear function of $\phi_1, \phi_2, \ldots \phi_n$ with constant coefficients cannot be zero unless all its coefficients are zero:

$$c_1(\alpha_{11} - \omega) + c_2 \alpha_{21} + \dots + c_n \alpha_{n1} = 0 \,,$$

$$c_1 \alpha_{12} + c_2(\alpha_{22} - \omega) \pm \dots + c_n \alpha_{n1} = 0 \,,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$c_1 \alpha_{1n} + c_2 \alpha_{2n} + \dots + c_n(\alpha_{nn} - \omega) = 0$$

from which

$$\begin{vmatrix} \alpha_{11} - \omega & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} - \omega & \dots & \alpha_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} - \omega \end{vmatrix} = 0.$$

This equation has the same roots as (10) and therefore coincides with it; it does not depend on the fundamental system from which it originates (because its roots are independent of it) and therefore its coefficients have been given the name of <u>invariants</u> of the differential equation.

(c) The determinant equation can be obtained by the formation of the fundamental equation, after setting $\omega = e^{2\pi i \rho}$ in it; however, $y = (x - x_0)^\rho$ can also be placed in the differential equation (1), then expanding the first member of the equation by means of increasing powers of $x - x_0$. When in this development a minimum power of $x - x_0$ is found, its coefficient, set equal to zero, will give an equation in ρ and, precisely, the determinant equation.

The formulas (16) are presented instead of (15) when the determinant equation has equal roots or roots differing from each other by integers.

(d) Results analogous to those of this Section and of the preceding one would be obtained, assuming that the variable rotates around a system of many singular points.

29. The operation by means of which, given a system of quantities ϕ_1, ϕ_2,

... ϕ_n, we obtain the quantities $\bar{\phi}_1$, $\bar{\phi}_2$,... $\bar{\phi}_n$ by using the relations /248

$$\begin{cases} \bar{\varphi}_1 = \alpha_{11}\varphi_1 + \alpha_{12}\varphi_2 + \dots + \alpha_{1n}\varphi_n \\ \bar{\varphi}_2 = \alpha_{21}\varphi_1 + \alpha_{22}\varphi_2 + \dots + \alpha_{2n}\varphi_n \\ \dots\dots\dots\dots \\ \bar{\varphi}_n = \alpha_{n1}\varphi_1 + \alpha_{n2}\varphi_2 + \dots + \alpha_{nn}\varphi_n \end{cases}$$

is called linear substitution; this operation is defined by the system of coefficients

$$\begin{array}{l} \alpha_{11}, \alpha_{22}, \dots, \alpha_{1n}, \\ \alpha_{21}, \alpha_{22}, \dots, \alpha_{2n}, \\ \dots\dots\dots \\ \alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nn}. \end{array} \qquad \text{(A)}$$

Since the determinant of the substitution $\Sigma \pm a_{11}a_{12}\dots a_{nn}$ is assumed to be different from zero, we can express φ_1,... ϕ_n by means of an analogous operation performed on $\bar{\phi}_1$, $\bar{\phi}_2$,... $\bar{\phi}_n$, and this operation is called inverse of the first.

The operation under discussion, which can be represented in abbreviated form with $A(\phi)$, is the element of a calculus called the calculus of linear substitutions; the inverse operation of A is represented by A^{-1}; if a second operation is defined by the system of coefficients

$$\begin{array}{l} \beta_{11}, \beta_{12}, \dots, \beta_{1n} \\ \beta_{21}, \beta_{22}, \dots, \beta_{2n} \\ \dots\dots\dots \\ \beta_{n1}, \beta_{n2}, \dots, \beta_{nn} \end{array} \qquad \text{(B)}$$

the operation which consists of executing first operation A, then B on ϕ_1, ϕ_2, ... ϕ_n is denoted by BA. It should be noted that AB is not generally equal to BA: when this is the case, (that they are mutually equal), the substitu- /249 tions A and B are said to be commutable. It is easy to verify that "the determinant of the product of two substitutions is equal to the product of the determinants of the single substitutions."

It is also immediately evident that if substitution A is applied to a system ϕ, so that ϕ is changed into $A(\varphi)$, a system $\psi = B(\phi)$, obtained from φ by means of a linear substitution B, is changed into $BAB^{-1}(\psi)$.

When an ensemble of operations in such that, by combining the operations therein contained in some way, operations of the same ensemble are always the result, it is said that that ensemble forms a group.

30. *Given a linear differential equation of the form (1); let x_0 be a non-singular point and ℓ a closed line and, departing from x_0, let us return to it without passing through any singular points. In the neighborhood of x_0 let $\phi_1, \phi_2, \ldots \phi_n$ be a fundamental system of integrals: this is such that ϕ_n will be a positive integer power series of $x - x_0$; if the analytic continuation of each of these series expansion is performed along line ℓ, it will return to point x_0 with an expansion equal to or different than that with which it had departed but which in any event can be put in the form

$$\overline{\varphi_h} = \alpha_{h1}\varphi_1 + \alpha_{h2}\varphi_2 + \cdots + \alpha_{hn}\varphi_n.$$

By following the line ℓ, a linear substitution A then performed on the fundamental system ϕ_h, thus a linear substitution corresponds to each line which leaves point x_0 and returns to it. Since the ensemble of lines leaving point

*For the concept of group of a differential equation and for its determination, see also the Memorandum by Fuchs (Crelle, Vols. 56 and 65) and Hamburger (op. cit., Vol. 73), those of Poincaré (Acta Matematica, Vols. 1 and 5, in various locations) especially that On the Groups of Linear Equations op. cit., Vol. 4 and Volterra (Memoranda of the Italian Society of Sciences, S. III, Vol. 6, and R. C. of the Circ. Mat. of Palermo, Vol. 2). See also Jordan, Cours d'Analyse, Vol. 3, p. 193.

x_0 evidently constitutes a group, and since the line composed of the two lines ℓ', ℓ'' successively traversed corresponds to the product BA of the linear substitutions corresponding to ℓ', ℓ'', it follows that the linear substitutions which can be performed on the fundamental system ϕ_h also constitute a group. This group is called the group of the differential equation. If instead of the fundamental system ϕ, another, ψ, is considered, since ψ is deduced from ϕ by means of a linear substitution $H(\phi)$, to the substitutions A acting on ϕ correspond the substitutions $HAH^{-1}(\psi)$ on ψ. The latter substitutions can be regarded as constituting a group which does not differ from the first because one and only one substitution of the one corresponds to each substitution /250 of the other and vice versa.

31. Let us consider a linear differential equation (1) with a finite number of singular points $z_1, z_2, \ldots z_p$ besides the point $z = \infty$ and with uniform coefficients. To each singular point z corresponds a system of canonic integrals (Sec. 27), for which the linear substitution which they undergo as a consequence of a rotation of the variable around z is

$$\begin{matrix} \omega_1 & 0 & 0 \ldots 0 \\ 0 & \omega_2 & 0 \ldots 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots \omega_n \end{matrix}$$

in the case that the fundamental equation has all its roots ω distinct, and the modification to be performed in the event that some of the roots ω are multiple is readily evident. Now the group of the differential equation (i.e., the substitutions undergone by a fundamental system for all the closed lines departing from the same point) will be found when we know the substitutions undergone by this fundamental system for the simple lines which, departing from any point x_o, make one rotation only around the r + 1 singular points, since every other closed path would lead to a combination or reiteration of these. These substitutions will then be known if the roots of the fundamental equations relative to 4 + 1 singular points as well as the relations which join the primitive fundamental system to the canonic integrals relative to each of the r + 1 singular points are known.

It should be noted that a simple line around $x = \infty$, described in a direction assumed to be positive, can be reduced to the succession of the simple lines described around the points $z_1, z_2, \ldots z_r$ in the negative direction: it follows from this that the said $S_1, S_2, \ldots S_r$, the substitutions undergone by the system ϕ through the rotations around $z_1, z_2, \ldots z_r$, and S_{r+1}, that undergone through a rotation around $x = \infty$, we will have:

$$S_1 S_2 \ldots S_r S_{r+1}(\varphi) = \varphi ,$$

or, symbolically, $S_1 S_2 \ldots S_r S_{r+1} = 1$.

CHAPTER V /251

REGULAR LINEAR DIFFERENTIAL EQUATIONS. - APPLICATION TO THE HYPERGEOMETRIC EQUATION

32. For applications of the preceding theory we must consider the equations (1) in which the coefficients $A_0, A_1, \ldots A_n$ are of the form

$$A_0 = P_0^n , A_1 = P_0^{n-1} P_1 , A_2 = P_0^{n-2} P_2 , \ldots A_n = P_n , \tag{A}$$

where $P_0, P_1, \ldots P_n$ are whole number rational polynomials in x of the respective ranks r, r-1, 2(r-1),...n(r-1). The roots of P_0 will be assumed to be distinct and be denoted by $z_1, z_2, \ldots z_r$. Such an equation (1) is said to be <u>regular</u>.

We do not exclude the fact that $P_1, P_2, \ldots P_n$ may be divisible by the same power of $x-z_1$; so that the equation (1) in which $A_0, A_1, \ldots A_n$ are polynomials of the respective ranks r, r-1,... r-n is of the indicated form, i.e., regular, as is immediately evident upon multiplying the whole equation by A_0^{n-1}.

33. Let us consider the integrals of this equation in the neighborhood of the singular point z_n and, for simplicity, we reduce this point to zero by putting $x' = x - z_h$. We endeavor to satisfy the equation by means of an expansion of the form

$$\varphi = x'^{\rho}(k_0 + k_1 x' + k_2 x'^2 + \ldots),$$

by applying the method of indeterminate coefficients; by that ϕ will be a canonic integral (Sec. 27). The term of the lowest rank in x' in the first member of the differential equation will be

$$k_0\left(a'_{0\cdot 0}\,\rho(\rho-1)\ldots(\rho-n+1) + a_{1\cdot 0}\,\rho(\rho-1)\ldots(\rho-n+2) + \ldots a_{n-1\cdot 0}\,\rho + a_{n\cdot 0}\right)x'^{\rho},$$

having been set

$$A_h = (a_{h\cdot 0} + a_{h\cdot 1}x' + \ldots a_{h\cdot n(r-1)}\,x'^{n(r-1)});$$

now since this term must be zero, the equation (determinant equation) /252

$$f_0(\rho) = a_{0\cdot 0}\,\rho(\rho-1)\ldots(\rho-n+1) + a_{1\cdot 0}\,\rho(\rho-1)\ldots(\rho-n+2) + \ldots + a_{n-1\cdot 0}\,\rho + a_{n\cdot 0} = 0, \tag{18}$$

should be satisfied and this will give us the values of ρ.

Now if equation (18) has neither multiple roots, nor roots differing from each other by whole numbers, one of these roots $\rho = \rho_1$ will be determined, and, through the same method of indeterminate coefficients, having set s = nr and

$$f_h(\rho) = a_{0\cdot h}\,\rho(\rho-1)\ldots(\rho-n+1) + a_{1\cdot h}\,\rho(\rho-1)\ldots(\rho-n+2) + \ldots + a_{n-1\cdot h}\,\rho + a_{n\cdot h},$$

we have the equations

$$
\begin{cases}
k_0 f_1(\rho_1) + k_1 f_0(\rho_1 + 1) = 0\,, \\
k_0 f_2(\rho_1) + k_1 f_1(\rho_1 + 1) + k_2 f_0(\rho_1 + 2) = 0\,, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
k_{\nu-s+n} f_{s-n}(\rho_1+\nu-s+n) + k_{\nu-s+n+1} f_{s-n-1}(\rho_1+\nu-s+n+1) + \ldots k_\nu f_0(\rho_1+\nu) = 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{cases}
\tag{19}
$$

by which the coefficients k_1, k_2,... k_ν,..., are determined, k_0 remaining arbitrary. The expansion ϕ, thus formally determined, is also convergent within a circle of center $x' = 0$ and with a radius which is not zero: this follows from the theorem of Sec. 10, and from the fact that the limit of $f_n(\rho_1+\nu)/f_0(\rho_1+\nu)$ for $\nu = \infty$ is finite.

34. Now we assume that the determinant equation has two equal roots or differing by a whole number μ. This is the same as supposing that the fundamental equation relative to the singular point x' has a double root, since the roots of the equations (18) are (cf. Secs. 27 and 28) the logarithms of the roots of the fundamental equation divided by $2\pi i$. In order to handle this case, it is convenient to premise the following two observations:

(a) An expression of the form

$$\mathfrak{P}(x) + \mathfrak{P}_1(x) \log \omega\,,$$

where $\mathfrak{P}(x)$, $\mathfrak{P}_1(x)$ are power series of x, cannot be zero in the neighborhood of $x = 0$ unless $\mathfrak{P}(x)$ and $\mathfrak{P}_1(x)$ and therefore all of their coefficients are zero. This immediately follows from the last proposition of Sec. 14 as can be seen by setting $x = e^{2\pi i \ell}$.

(b) By deriving the expression $x^\rho \log x$ n times with respect to x, we /253
obtain

$$\frac{\partial^n x^\rho \log x}{\partial x^n} = \rho(\rho-1)\dots(\rho-n+1)x^{\rho-n}\log x + x^{\rho-n}\frac{\partial \rho(\rho-1)\dots(\rho-n+1)}{\partial \rho}:$$

this formula is verified for $n = 2$, and it is readily evident that if it is true for a given n, it holds also for $n + 1$.

Having established this, the general theory (Sec. 27) shows us that when the fundamental equation has a double root, an integral of the differential equation which contains two arbitrary constants exists and has the form,

$$\varphi = vx'^\rho + v_1 x'^\rho \log x',$$

where ν and ν_1 are functions to a value of x' around $x' = 0$.

If we now take a positive and whole number power series of x for ν and ν_1,

$$v = \sum_{\nu=0}^{\infty} k_\nu x^\nu \quad . \quad v_1 = \sum_{\nu=0}^{\infty} h_\nu x^\nu,$$

and we substitute the expression ϕ in the equation (1), we shall have, in accordance with the observations (a) and (b), the following two systems of equations in which $f_h'(\rho)$ represents the derivative of $f_h(\rho)$ with respect to ρ:

$$\left\{\begin{array}{l} k_0 f_0(\rho) + h_0 f_0'(\rho) = 0 \\ k_0 f_1(\rho) + k_1 f_0(\rho+1) + h_0 f_1'(\rho) + h_1 f_0'(\rho+1) = 0, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ k_{\nu-s+n} f_{s-n}(\rho+\nu-s+n) + k_{\nu-s+n+1} f_{s-n-1}(\rho+\nu-s+n+1) + \dots + \\ + k_\nu f_0(\rho+\nu) + h_{\nu-s+n} f'_{s-n}(\rho+\nu-s+n) + \dots + h_\nu f_0'(\rho+\nu) = 0, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{array}\right. \qquad (20)$$

and

$$
\begin{cases}
h_0 f_0(\rho) = 0\,, \\
h_0 f_1(\rho) + h_1 f_0(\rho + 1) = 0\,, \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
h_{\nu-s+n} f_{s-n}(\rho+\nu-s+n) + h_{\nu-s+n+1} f_{s-n-1}(\rho+\nu-s+n+1) + \ldots + h_\nu f_0(\rho+\nu) = 0 \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{cases}
\tag{21}
$$

In the event that the determinant equation $f_0(\rho) = 0$ has two equal roots, /254
system (21) gives us the values of $h_1, h_2, \ldots h_\nu, \ldots$, h_0 being arbitrary; then, the first equation of (20) being identically satisfied, an arbitrary value is given to k_0 and $k_1, k_2, \ldots$ are determined as a function of the values already obtained for the h. In the event, however, that the equation $f_0(\rho) = 0$ has two roots differing by a whole number μ (taken to be positive), the first equation of (20) no longer is identically satisfied and it should be true that $h_0 = h_1 = \ldots h_{\mu-1} = 0$; the h_μ is arbitrary and the equations (21), from the $\mu + 2^{nd}$ on, determine $h_{\mu+1}, h_{\mu+2}, \ldots$ The equations (20) determine $k_1, k_2, \ldots$ up to $k_{\mu-1}$ as a function of the arbitrary k_0; k is determined as a function of this and of the arbitrary h_μ, and the following equations (20) determine $k_{\mu+1}, k_{\mu+2}, \ldots$ by means of substituting in it the values furnished by the equations of (21) for $h_{\mu+1}, h_{\mu+2}, \ldots$ The integral ϕ thus has its coefficient determined with two arbitrary constants; the convergence of the power series ν and ν_1 is verified within a non-zero circle on the basis of the theorem in Sec. 10 for the series ν_1 and on the basis of the analogous theorem in Sec. 11 for the series ν.

An analogous procedure is followed in the event that the fundamental equation has a triple root: we will obtain in a similar way an integral of (1), containing three arbitrary constants, having the form

$$\varphi = v x'^{\rho} + v_1 x'^{\rho} \log x' + v_2 x'^{\rho} \log^2 x',$$

where ν, ν_1, and ν_2 are positive integer power series of x' converging in a given neighborhood of $x' = 0$: and this holds both when the corresponding roots of the determinant equation are equal and when they differ by whole numbers. An analogous situation prevails for roots of the fundamental equation having any order of multiplicity.

It is hardly necessary to point out that, having demonstrated the convergence of the expansion in the preceding power series in a circle of center $x' = 0$ (or $x = z$) and of non-zero radius, it follows from the principles of the theory of functions that the radius of convergence of the above mentioned series will be at least equal to the distance of the point z_k from the nearest of the other singular points.

Analogous considerations hold for the point $x = \infty$, as it is evident from the transformation $x = 1/z$, and the same results are maintained, provided that the positive integer power series of x are substituted with negative integer power series of the same variable. If we put z_{r+1} in place of ∞, and substitute, as usual, $1/x$ for $x - \infty$, and, lastly, if we use the symbol $\mathfrak{P}(x)$ to represent a positive and integer power series of x, we can state the following result:

"A regular linear differential equation with singular points z_h, ($h = 1, 2, 3, \ldots r, r + 1$) in the neighborhood of each singular point and, in correspondence to the simple roots ρ of the determinant equation relative to that point, has canonic integrals of the following form /255

$$(x - z_h)^{\rho}\, \mathfrak{P}(x - z_h). \tag{22}$$

The presence of a multiple root ρ of the determinant equation leads to an integral of the preceding form and also to integrals of the form

$$\begin{cases} (x-z_h)^{\rho}\left(\mathfrak{P}(x-z_h) + \mathfrak{P}_1(x-z_h)\log(x-z_h)\right) \\ (x-z_h)^{\rho}\left(\mathfrak{P}(x-z_h) + \mathfrak{P}_1'(x-z_h)\log(x-z_h) + \mathfrak{P}_2'(x-z_h)\log^2(x-z_h)\right) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases} \tag{23}$$

up to an integral containing $\log^{\lambda}(x-z_h)$ if the root ρ and of the order λ of multiplicity."

35. By denoting with $\rho_{h.k}(k = 1, 2, 3, \ldots n)$ the roots of the fundamental equation relative to the point $z_h(h = 1, 2, \ldots r, r+1)$ assumed to be simple, the integrals of the form (22) relative to that point will form a fundamental system which we shall designate by ψ_h. By representing with S_h the linear substitution undergone by ψ_h through a rotation of the variable around z_h, the determinant of the substitution will obviously be

$$|S_h| = e^{2\pi i \sum_{k=1}^{n} \rho_{h\cdot k}}.$$

Now, each fundamental system ϕ can be put in the form $T_h\psi_h$, T_h being the symbol of a convenient linear substitution, and the substitution undergone by ϕ through a rotation of the variable around z_h is $T_hS_hT_h^{-1}$ (Sec. 29). Recalling now the relation established at the conclusion of Sec. 31 between the substitutions undergone by a system ϕ in the rotations of the variable around the $r + 1$ singular points, we have

$$T_1 S_1 T_1^{-1} T_2 S_2 T_2^{-1} \ldots T_{r+1} S_{r+1} T_{r+1}^{-1} = 1,$$

and since the determinant of the substitution product is equal to the product of the determinants of the single substitutions, (Sec. 29) we have

$$|S_1||S_2|\dots|S_{r+1}|=1,$$

whence

$$e^{2\pi i\sum_{h=1}^{r+1}\sum_{k=1}^{n}\rho_{h,k}}=1.$$

and therefore the sum of the $\rho_{h.k}$ will be a whole number. However, the ex- /256
pressions (22) are not changed if the ρ varies by whole numbers, therefore, we can write without restriction:

$$\sum_{h=1}^{r+1}\sum_{k=1}^{n}\rho_{h,k}=1. \tag{24}$$

36. The regular differential equation has its coefficients formed by the polynomials P_0, P_1,... P_n (Sec. 32) of the ranks r, r - 1, 2(r - 1),... n(r - 1). Given the singular points z_1, z_2,... z_r, roots of P_0, the equation contains the coefficients of P_1, P_2,... P_n, in the amount of

$$\frac{rn(n+1)-n(n-1)}{2}$$

unknowns; on the other hand, the n(r + 1) exponents $\rho_{h.k}$, being related by the relation (24), constitute a system of n(r + 1) - 1 unknowns. The number of the first is generally greater than that of the second; the equality condition

$$\frac{rn(n+1)-n(n-1)}{2}=n(r+1)-1$$

easily leads, as is evident by resolving with respect to n and making an abstraction of the value n = 1, to the equation

$$n=\frac{2}{r-1},\quad \text{whence } n = 2,\ r = 2$$

From that we conclude that:

"A regular linear differential equation of an order greater than the first order is not generally determined by the knowledge of its singular points and of the exponents in these points. However, the equation of the second order with two singular points at a finite distance is completely determined by the knowledge of these numbers."

37. The regular equation of the second order which we have obtained in the preceding Section will have the form

$$P_0^2 y'' + P_0 P_1 y' + P_2 y = 0,$$

being

$$P_0 = (x - z_1)(x - z_2)$$

and P_1 and P_2 whole number rational polynomials of the first and second order respectively. Meanwhile, an easy linear transformation of the variable permits the singularity of the equation to be brought into the point 0 and into the point 1, ($z_1 = 0$, $z_2 = 1$); thus, the equation under discussion can be written /257

$$x^2(x-1)^2 y'' + x(x-1)(hx + h') y' + (gx^2 + g'x + g'') y = 0, \tag{25}$$

where the coefficients h, h', g, g', g" have to be determined as a function of the exponents $\rho_{h \cdot k}$ relative to the singular points x = 0, x = 1, x = ∞. Our purpose now is to determine them and study the properties of the integrals of the equation (25); then to simplify the form of the equation itself with the use of these properties.

For this purpose we note primarily that if we set

$$y = x^p (x-1)^q v, \tag{26}$$

the function ν will satisfy an equation of the same form (25), which can be readily verified with the simple substitution. By denoting the exponents relative to the point x = ∞ with ρ and ρ', those relative to x = 0 with ρ_0 and ρ_0', those relative to x = 1 with ρ_1 and ρ_1' (excluding the case in which ρ and ρ' are equal or different by whole numbers and thus for ρ_0 and ρ_0' and for ρ_1 and ρ_1') the substitution (26) will transform these pairs of exponents into

$$\rho+p\ ,\ \rho'+p\ \ ;\ \ \rho_0-p\ ,\ \rho_0'-p\ \ ;\ \ \rho_1-q\ ,\ \rho_1'-q$$

respectively, so that "the transformations of the form (26) leave unchanged the differences of the exponents $\rho - \rho'$, $\rho_0 - \rho_0'$, $\rho_1 - \rho_1'$," furthermore, p and q can be chosen in such a way that for the new equation (25) two of the exponents not relative to the same point, e.g., ρ_0' and ρ_1', are equal to zero.

Having established this, the determinant equations of equation (25) relative to the points x = ∞, x = 0 and x = 1 are calculated without difficulty (Sec. 33) and are, respectively

$$\rho(\rho+1)-h\rho+g=0,$$

$$\rho(\rho-1)-h'\rho+g''=0,$$

$$\rho(\rho-1)+(h+h')\rho+(g+g'+g'')=0;$$

and among these, the first has for roots ρ and ρ', the second ρ_0 and ρ_0', the third ρ_1 and ρ_1'. If we now suppose, in accordance with what has been said, equation (25) to be reduced so that ρ_0' and ρ_1' are equal to zero, we shall /258 have

$$g''=0\ \ ,\ \ g+g'+g''=0\,,$$

or

$$g''=0\ \ ,\ \ g=-g'.$$

By putting α and β in place of ρ, ρ', the first of the preceding determinant equations gives us

$$h=\alpha+\beta+1\ \ ,\ \ g=-g'=\alpha\beta\,;$$

then putting $\rho_0 = 1 - \gamma$, we have $h' = -\gamma$, whence $\rho_1 = \gamma - \alpha - \beta$; (24) is thus verified by these expressions.

By substituting in equation (25) for h, h', g, g', g'' their values and reducing, we obtain the same equation in the form:

$$x(x-1)y'' + \left((\alpha+\beta+1)x - \gamma\right)y' + \alpha\beta y = 0. \tag{27}$$

This is given the name of <u>hypergeometric differential equation</u>; its general integral is called a <u>hypergeometric function</u>, a particular integral, a <u>branch</u> of the function. It should be recalled that the substitutions

$$x = \frac{1}{z}\ ,\quad x = 1 - z\ ,\quad x = 1 - \frac{1}{z}\ ,\quad x = \frac{z}{z-1}\ ,\quad x = \frac{1}{1-z},$$

which <u>form a group</u>*, transform (25) into an equation of the same form in which exchange of the exponent pairs is the only modification.

38. Whenever, by following the method in Sec. 33, we wish to obtain the expansions of the canonic integrals of (27) around the singular points 0, 1, ∞, we shall easily find that these expansions are expressed by means of hypergeometric series. In the neighborhood of the point $x = 0$, the exponents are $1 - \gamma$ and 0, and the corresponding canonic integrals are

$$x^{1-\gamma} F(\alpha + 1 - \gamma\ ,\ \beta + 1 - \gamma\ ,\ 2 - \gamma\ ,\ x) \quad \text{ed} \quad F(\alpha, \beta, \gamma, x);$$

(the fact that this last series is an integral of (27) has already been noted as early as Sec. 4, c). The exponents are $\gamma - \alpha - \beta$ and 0 for the point $x = 1$, and the corresponding integrals are /259

$$(x-1)^{\gamma-\alpha-\beta} F(\gamma - \beta\ ,\ \gamma - \alpha\ ,\ \gamma - \alpha - \beta + 1\ ,\ 1 - x) \quad \text{ed} \quad F(\alpha\ ,\ \beta\ ,\ \alpha + \beta + 1 - \gamma\ ,\ 1 - x),$$

---

*Called an <u>anharmonic group</u> for the well-known relationships between the anharmonic ratios which give rise to 4 elements in a geometric form of the first species.

finally, the exponents are α and β for the point $x = \infty$, and the corresponding integrals are

$$x^{-\alpha} F\left(\alpha, \alpha-\gamma+1, \alpha-\beta+1, \frac{1}{x}\right) \text{ ed } x^{-\beta} F\left(\beta, \beta+\gamma+1, \beta-\alpha+1, \frac{1}{x}\right).$$

Turning now to the Gauss series, the properties of which are summarized in Chapter I, we recall that this series is convergent for $|x| < 1$ and possibly for points of the circumference $|x| = 1$ but never for $|x| > 1$ if the values of the parameters are finite. The analytic function represented by this series has an analytic continuation which always satisfies equation (27) and is therefore a hypergeometric function; however, in order to know the value that the function assumes when the analytic continuation is performed in accordance with a determinate line (not passing through the points 0, 1 and ∞), it is necessary to know the group of the differential equation as defined in Sec. 27.

39. The determination of the group of equation (25) does not present greater difficulties than for equation (27) and therefore will be performed through the former. We shall denote with u, u', with v, v' and with w, w' the pairs of canonic integrals relative to the points 0, 1 and ∞. The group of the equation will be completely determined if we know the substitutions undergone by u, u' for the three simple lines which, departing from a point $x_0(|x_0| < 1)$, surround only one of the points 0, 1 and ∞, since any other closed path would lead to a combination or reiteration of these simple lines. We shall denote with A, B and C the substitutions relative to these three lines.

The substitution A is known because after one rotation of x around 0, u and u' are changed into $ue^{2\pi i\rho_0}$ and $u'e^{2\pi i\rho'_0}$, as are also known the analogous substitutions undergone by v, v' during a rotation around 1 and by w and w' during a rotation around ∞.

Now setting

$$\begin{cases} u = \alpha_{11} v + \alpha_{12} v' \\ u' = \alpha_{21} v + \alpha_{22} v' \end{cases}$$

and

$$\begin{cases} u = \beta_{11} w + \beta_{12} w' \\ u' = \beta_{21} w + \beta_{22} w' \end{cases}$$

or symbolically, (u) = S(v) and (u) = S'(w), it is evident that if the substitutions S and S' were known, the A, B and C and thereby the required group of the equation (27) would be immediately deduced from them. /260

In order to determine S and S', we observe that a rotation in the positive direction around x = 1 is equivalent to a rotation in the negative direction around x = ∞ followed by a negative turn around x = 0, as also can be deduced from the relation of Sec. 31, which in our case is written ABC = 1.

Therefore, we shall have

$$\alpha_{11} e^{2\pi i\rho_1} v + \alpha_{12} e^{2\pi i\rho_1'} v' = e^{-2\pi i\rho_0} (\beta_{11} e^{-2\pi i\rho} w + \beta_{12} e^{-2\pi i\rho'} w'),$$

and because

$$\alpha_{11} v + \alpha_{12} v' = \beta_{11} w + \beta_{12} w',$$

we have, by eliminating v',

$$\alpha_{11} (e^{2\pi i\rho_1} - e^{2\pi i\rho_1'}) v = \beta_{11} (e^{-2\pi i(\rho_0+\rho)} - e^{-2\pi i\rho_1'}) w + \beta_{12} (e^{-2\pi i(\rho_0+\rho')} - e^{-2\pi i\rho_1'}) w'.$$

Analogously we find

$$\alpha_{21}(e^{2\pi i\rho_1} - e^{2\pi i\rho_1'}) v = \beta_{21}(e^{-2\pi i(\rho_0'+\rho)} - e^{-2\pi i\rho_1'}) w + \beta_{22}(e^{-2\pi i(\rho_0'+\rho')} - e^{-2\pi i\rho_1'}) w'$$

however, these linear relations between v, w and w' cannot be distinct, otherwise w = kw' would be deduced, which cannot be; therefore, the coefficients of the preceding relations should be proportional, and we have

$$\frac{\alpha_{11}}{\alpha_{21}} = \frac{\beta_{11}}{\beta_{21}} \frac{e^{-2\pi i(\rho_0+\rho)} - e^{-2\pi i\rho_1'}}{e^{-2\pi i(\rho_0'+\rho)} - e^{-2\pi i\rho_1'}} = \frac{\beta_{12}}{\beta_{22}} \frac{e^{-2\pi i(\rho_0+\rho')} - e^{-2\pi i\rho_1'}}{e^{-2\pi i(\rho_0'+\rho')} - e^{-2\pi i\rho_1'}}.$$

Analogously, we obtain, by eliminating v:

$$\frac{\alpha_{12}}{\alpha_{22}} = \frac{\beta_{11}}{\beta_{21}} \frac{e^{-2\pi i(\rho_0+\rho)} - e^{-2\pi i\rho_1}}{e^{-2\pi i(\rho_0'+\rho)} - e^{-2\pi i\rho_1}} = \frac{\beta_{12}}{\beta_{22}} \frac{e^{-2\pi i(\rho_0+\rho')} - e^{-2\pi i\rho_1}}{e^{-2\pi i(\rho_0'+\rho')} - e^{-2\pi i\rho_1}},$$

and from these equations we can determine the ratios

$$\frac{\alpha_{11}}{\alpha_{21}},\ \frac{\alpha_{12}}{\alpha_{22}},\ \frac{\beta_{11}}{\beta_{21}},\ \frac{\beta_{12}}{\beta_{22}}$$

as a function of one of them, and, inasmuch as u, u', v, v', w and w' are determinate except for one arbitrary constant multiplier, the substitutions S and S' are thus determined.

It should be noted that the determination of the preceding ratios remains the same if the exponents ρ, ρ',... vary by integers; it follows that

"Those hypergeometric functions whose parameters differ by whole numbers have the same group of substitutions."

40. We should now like to demonstrate that

"If three equations of the form (25) are such that the exponents in the points 0, 1 and ∞ differ by whole numbers, a linear relation with rational coefficients exists among the integrals of these equations."

For this purpose let us denote the canonic integrals of the first equation relative to the point x = 0 by u and u', those relative to the point x = 1 by v and v', those relative to x = ∞ by w and w'; then, the analogous integrals for the second equation by u_1 and u_1', v_1 and v_1', w_1 and w_1', and the analogous integrals for the third by u_2 and u_2', v_2 and v_2', w_2 and w_2'.

Then we form the sum of the exponents relative to the point $x = 0$ for the first, second and third equations and denote the smallest of these three sums by σ_0; thus σ_1 would be the smallest sum relative to $x = 1$, and σ the smallest sum relative to $x = \infty$.

It will evidently suffice to demonstrate the theorem for a branch of each general integral, e.g., through u, u_1 and u_2.

In this event, considering the determinant to be $D = u_1'u_2 - u_2'u_1$, this gives an analytic function which has the form of $x^{\sigma_0}\mathfrak{P}(x)$ in the neighborhood of $x = 0$. By expressing u and u' as a function of v and v'

$$\begin{cases} u = \alpha_{11} v + \alpha_{12} v' \\ u' = \alpha_{21} v + \alpha_{22} v' \end{cases}$$

the same transformation holds for u_1, u_1' and u_2, u_2' as a function of v_1, v_1' and v_2, v_2', inasmuch as the three equations have the same group; thus the function D in the neighborhood of $x = 1$ will have the form

$$(\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})\, x^{\sigma_1}\, \mathfrak{P}_1(x-1),$$

so that the function

$$G = Dx^{-\sigma_0}(x-1)^{-\sigma_1}$$

is regular over the entire plane except when $x = \infty$: thus, according to the principles of the theory of functions, it is a rational or trascendental integer function. However, when $x = \infty$, D has the form $x^{-\sigma}\mathfrak{P}_2(1/x)$, whence /262

$$G = x^{-(\sigma_0+\sigma_1+\sigma)}\,\mathfrak{P}_3\left(\frac{1}{x}\right);$$

and $\sigma_0 + \sigma_1 + \sigma$ being finite, G will be rational and an integer (and remains so, having independently shown that $\sigma_0 + \sigma_1 + \sigma$ is a whole number). Thus

$$u_1' u_2 - u_2' u = x^{\sigma_0}(x-1)^{\sigma_1} G.$$

Analogously

$$u_2' u - u' u_2 = x^{\sigma_0}(x-1)^{\sigma_1} G_2 \quad , \quad u' u_1 - u u_1' = x_0^{\sigma_0}(x-1)^{\sigma_1} G_3 ,$$

G_1 and G_2 also being whole number rational polynomials. It follows from this that the identically zero determinant

$$\begin{vmatrix} u'' & u_1'' & u_2'' \\ u' & u_1' & u_2' \\ u & u_1 & u_2 \end{vmatrix}$$

can be written

$$G u + G_1 u_1 + G_2 u_2 = 0 ,$$

q.e.d.

The relation among the contiguous hypergeometric functions is a special case of the theorem now demonstrated*.

CHAPTER VI /263

A FUNCTIONAL TRANSFORMATION. ITS APPLICATIONS TO THE GENERALIZATION OF HYPERGEOMETRIC FUNCTIONS ACCORDING TO POCHHAMMER AND GOURSAT.

41. In a regular and linear differential equation

$$A_0 \varphi^{(n)} + A_1 \varphi^{(n-1)} + \ldots + A_n \varphi = 0$$

the rank of the polynomials A_0, $A_1 \ldots A_n$ decrease in order of one unit and therefore the rank of A_0 can not be less than n. If it is greater, and equal to n + p, we can set $\phi = \psi^{(p)}$: the equation assumes the order n + p and in it

*Cf. Riemann, Werke, p. 67.

the rank of each coefficient is equal to the index of the derivative that it multiplies. Therefore, either the regular equation is such that the coefficient of $\phi^{(h)}$ in it is of the rank <u>h</u>, and in this case we shall call it <u>normal</u>, or it is brought back to this case by means of the indicated setting.

We write the <u>normal</u> equation in the form

$$B_n \varphi^{(n)} + B_{n-1} \varphi^{(n-1)} + \ldots + B_0 \varphi = 0. \tag{28}$$

where each polynomial B_h is of the rank indicated by its subscript; the first member of this equation is called the <u>normal linear differential form</u> of the order n, and the two following operations are defined on this form. The first, which will be represented by D, consists of deducing from the form (which we shall denote by $\Delta(\phi)$ or simply by Δ) the new form:

$$D\Delta = B_n' \varphi^{(n-1)} + B'_{n-1} \varphi^{(n-2)} + \ldots + B_1' \varphi,$$

where B_h' is the derivative of B_h; this form is also normal, but of the order n - 1; by repeating operation D, the new normal forms of the orders of n - 2, n - 3, ... will be obtained

$$D^2\Delta = B_n'' \varphi^{(n-2)} + B_{n-1}'' \varphi^{(n-3)} + \ldots + B_2'' \varphi,$$

$$D^3\Delta = B_n''' \varphi^{(n-3)} + B_{n-1}''' \varphi^{(n-4)} + \ldots + B_3''' \varphi, \ldots$$

The second operation, which will be represented by S_σ, is defined by /264

$$S_\sigma\Delta = \Delta + \sigma D\Delta + \binom{\sigma}{2} D^2\Delta + \ldots + \binom{\sigma}{n} D^n\Delta,$$

where $\binom{\sigma}{h}$ represents, as usual, the binomial coefficient $\frac{\sigma(\sigma-1)\ldots(\sigma-h+1)}{1\cdot 2\cdot 3\ldots h}$.

It is immediately verified that $S_\sigma\Delta$ is a normal form of the order n, which can also be written

$$S_\sigma \Delta = B_n \varphi^{(n)} + (B_{n-1} + \sigma B_n')\varphi^{(n-1)} + \left(B_{n-2} + \sigma B'_{n-1} + \binom{\sigma}{2} B_n''\right)\varphi^{(n-2)} + \ldots$$

$$\ldots + \left(B_0 + \sigma B_1' + \binom{\sigma}{2} B_2'' + \ldots + \binom{\sigma}{n} B_n^{(n)}\right)\varphi.$$

The operations thus defined both clearly possess the distributive property. They are also mutually commutable, which can be readily verified. Finally, a simple calculation based on the known property of binomial coefficients allows the following relation to be demonstrated

$$S_\sigma S_{\sigma'} \Delta = S_{\sigma+\sigma'} \Delta;$$

so that the quantity σ behaves like an exponent in the operative symbol S. Particularly, $S_0\Delta = \Delta$, or symbolically, $S_0 = 1$, and if a form Δ_1 is deduced from Δ by means of the operation S_0, inversely, Δ will be deduced from Δ_1 by means of the operation $S_{-\sigma}$, or symbolically, $S_\sigma S_{-\sigma} = 1$*.

42. Let us consider the expression in which σ is any number which is neither a positive integer nor zero:

$$\psi(x) = \int_{(\lambda)} \frac{\varphi(t)\, dt}{(t-\lambda)^{\sigma+1}}, \quad (^{**}) \tag{29}$$

where the integration is extended to a line λ so that the second member has meaning and that the integration by parts and the derivation under the sign are admissible. The expression /265

$$\int_{(\lambda)} \frac{B_1(t)\varphi(t)\, dt}{(t-x)^{\sigma+1}}$$

will now be able to be expressed as follows:

---

*See my memorandum On Linear Differential Forms. (R. C. della R. Accademia dei Lincei, May 8, 1892).

**For integrals of this form, see Pochhammer, Crelle, Vol. CIV. cf. one of my works in Mem. Acad. Bologna, Vol. II, S. V.

Being identically

$$B_1(l) = B_1(x) + B_1'(l - x),$$

it will be

$$\int_{(\lambda)} \frac{B_1(l)\,\varphi'(l)\,dl}{(l - x)^{\sigma+1}} = B_1(x)\int_{(\lambda)} \frac{\varphi'(l)\,dl}{(l - x)^{\sigma+1}} + B_1'\int_{(\lambda)} \frac{\varphi'(l)\,dl}{(l - x)^{\sigma}};$$

now for the first of these integrals, we have, by deriving (28), then integrating by parts

$$\int_{(\lambda)} \frac{\varphi'(l)\,dl}{(l - x)^{\sigma+1}} = \psi'(x) + \left[\frac{\varphi(l)}{(l - x)^{\sigma+1}}\right]_\lambda$$

for the second, integrating by parts

$$\int_{(\lambda)} \frac{\varphi'(l)\,dl}{(l - x)^{\sigma}} = \left[\frac{\varphi(l)}{(l - x)^{\sigma}}\right]_\lambda + \sigma\int_{(\lambda)} \frac{\varphi(l)\,dl}{(l - x)^{\sigma+1}};$$

so that

$$\begin{aligned}\int_{(\lambda)} \frac{B_1(l)\,\varphi'(l)\,dl}{(l - x)^{\sigma+1}} &= B_1(x)\,\psi'(x) + \sigma B_1'\,\psi(x) + L_1\\ &= S_\sigma B_1(x)\,\psi'(x) + L_1.\end{aligned}$$

L_1 being an expression to the limits of the integration, containing $\phi(\ell)$ linearly.

Analogously we find

$$\int_{(\lambda)} \frac{B_2(l)\,\varphi''(l)\,dl}{(l - x)^{\sigma+1}} = S_\sigma B_2(x)\,\psi''(x) + L_2$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\int_{(\lambda)} \frac{B_n(l)\,\varphi^{(n)}(l)\,dl}{(l - x)^{\sigma+1}} = S_\sigma B_n(x)\,\psi^{(n)}(x) + L_n,$$

where L_h is an expression to the limits of the integration, which contains linearly

$$\varphi(l),\ \varphi'(l), \dots \varphi^{(n-1)}(l).$$

From these, due to the distributive property of equation S, it follows /266 that

$$\int_{(\lambda)} \frac{\Delta(\varphi)\,dl}{(l-x)^{\sigma+1}} = S_\sigma \Delta(\psi) + L \tag{30}$$

where L is an expression of the same form as L_h, which contains linearly $\phi(\ell)$, $\phi'(\ell), \dots \phi^{(n-1)}(\ell)$ to the limits of the integration*.

The line λ of integration may be chosen so that the part L to the limits is zero; this can be done either by taking for λ an open line and such that $\phi(\ell)$, $\phi'(\ell), \dots \phi^{(n-1)}(\ell)$ are zero at their ends or by taking for λ a closed line not containing x and such that, the variable having traversed its path and returned to the point of departure (beginning), $\phi(\ell)$ and its first n - 1 derivatives have the same value. In this hypothesis, (30) becomes

$$\int_{(\lambda)} \frac{\Delta(\varphi)\,dl}{(l-x)^{\sigma+1}} = S_\sigma \Delta(\psi), \tag{30'}$$

and the following proposition can be stated:

"If ϕ is an integral of the linear differential equation $\Delta = 0$, the expression (29) will give us an integral ψ of the transformed equation $S_\sigma\Delta = 0$, the line λ being chosen so that the part L at the limits is annulled. The integral ψ contains the same number of arbitrary constants of ϕ and therefore will be the general integral of the equation $S_\sigma\Delta = 0$ if ϕ is the general integral of $\Delta = 0$".

We may add, recalling the property $S_{\sigma+\sigma'} = S_\sigma S_{\sigma'}$, that

---

*This formula is demonstrated by the case in which σ is not a positive whole number. If σ is such, we arrive at the same formula, but the procedure in the demonstration is subject to slight but obvious modifications.

"The integration of an equation $S_\sigma\Delta = 0$ for a special value of σ leads to the integration of the equation itself for every other value of σ."

43. The following observations can be added here:

(a) The property $S_\sigma S_{-\sigma} = 1$ immediately permits of transposing the definite integral (29), i.e., of expressing ϕ by means of an integral containing ψ under the sign.

(b) All the equations $S_\sigma\Delta = 0$ are regular, normal and have the same singular points. Since $f_1(\rho) = 0$ is the determinant equation of $\Delta = 0$ relative to a singular point, it is immediately evident that the determinant equation of $S_\sigma\Delta = 0$ relative to the same point is $f(\rho+\sigma) = 0$.

(c) The equations $S_\sigma\Delta = 0$ relative to different values of σ differing from each other by integers have the same group and therefore n + 1 integrals of such equations are related by a linear relation with rational coefficients. /267 (The demonstration "in extenso" of this theorem is left to the reader, a demonstration perfectly analogous to that given in Sec. 40.

44. It follows from the theorem of Sec. 42 that whenever the integration of an equation $S_\sigma\Delta = 0$ for a special value σ_0 of σ is known, the integral of the equation for every other value of σ can be expressed through a definite integral in which the integral of $S\sigma_0\Delta = 0$ is involved under the sign. Then we are able to give as many different applications of this theorem as there are cases in which we know how to integrate the equation for a special value of σ. In the following, two important applications of this method will be given; one of them, in this and in the three subsequent sections, will acquaint us with the theory of the generalization of the hypergeometric equation given by Pochhammer*; the second deals with the generalization which, in a different direc-

*Crelle, Vol. LXXI, cf. Jordan, Cours d'Analyse, 1st edition, Vol. III, p. 241; Goursat, Acta Mathematica, Vol. II.

tion, Goursat has given for the hypergeometric equation* and is treated in Sec. 40.

In the first of these applications, we assume that one of the forms $S_\sigma\Delta$, (and it can be assumed without restriction that this expression is relative to $\sigma = 0$, i.e., Δ itself) is reduced to its first two terms. We then set

$$\Delta = B_n \varphi^{(n)} + B_{n-1} \varphi^{(n-1)}$$

then integrating (28) by parts $n - 1$ times, always assuming λ such that the part at the limits is zero, we have:

$$\psi(x) = \frac{1}{\sigma(\sigma-1)\ldots(\sigma-n+1)} \int_{(\lambda)} \frac{\varphi^{(n-1)}(t)\,dt}{(t-x)^{\sigma-n+2}}. \tag{30}$$

Now setting $\Delta = 0$, $\varphi^{(n-1)}$ will be the integral of an equation of the first order and its expression can be given explicitly; (30) will then give us the expression of the integral of $S_\sigma\Delta = 0$. Whence the following result:

"The integral of the linear differential equation by Pochhammer $S_\sigma\Delta = 0$ can be obtained in the form of a definite integral, or by expanding

$$B_n\psi^{(n)} + \left(B_{n-1} + \sigma B_n'\right)\psi^{(n-1)} + \left(\sigma B'_{n-1} + \binom{\sigma}{2} B_n''\right)\psi^{(n-2)} + \ldots$$
$$\ldots + \left(\binom{\sigma}{n-1} B_{n-1}^{(n-1)} + \binom{\sigma}{n} B_n^{(n)}\right)\psi = 0. \text{»} \tag{31}$$

In order to complete this treatment, we must explicitly give the form of ψ, determine the integration lines λ satisfying the imposed conditions, and, finally, indicate how the group of the equation (31) can be obtained. /268

45. (a) Thus we have the equation

$$B_n f' + B_{n-1} f = 0 \quad , \quad f = \varphi^{(n-1)}$$

*Annales de l'Ecole Normale, S. II, Vol. III.

and we set

$$B_n = (t - a_1)(t - a_2)\dots(t - a_n),$$

limiting ourselves for brevity to the case in which the equation $B_n(\ell) = 0$ has simple roots*. We obtain, a_1, a_2,.. a_n being constants which can be determined:

$$\frac{f'}{f} = -\frac{B_{n-1}}{B} = \frac{\alpha_1}{t - a_1} + \frac{\alpha_2}{t - a_2} + \dots + \frac{\alpha_n}{t - a_n},$$

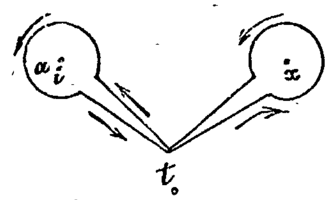
whence, c being an arbitrary constant:

$$f(t) = c(t - a_1)^{\alpha_1}(t - a_2)^{\alpha_2}\dots(t - a_n)^{\alpha_n},$$

so that the integral of the equation (31) is put in the form

$$\psi(x) = c\int_{(\lambda)} \frac{(t - a_1)^{\alpha_1}(t - a_2)^{\alpha_2}\dots(t - a_n)^{\alpha_n}\,dt}{(t - x)^{c-n+2}}. \qquad (32)$$

(b) We shall now determine the integration line λ, which should satisfy the conditions stated in Sec. 42. Thus, we denote with ℓ_i (i = 1, 2, 3... n) a line which, leaving any point ℓ_0 of the plane of the complex variable $\mathcal{L}$, approaches point α_i, goes around this point,



remaining close to it, and then returns to the point of departure ℓ_0, without including any of the other points a_1, a_2,... nor the point x and without passing through any of these points, then we denote with ℓ_x an analogous line which, upon leaving ℓ_0, returns there after having included only the point x. /269

*The case in which the equation B = 0 has multiple roots would require a somewhat more involved procedure but not different concepts. For the treatment of this case, v. p. e. Jordan, Cours d'Analyse, 3: from p. 241 on.

These lines are understood traversed in the direction which we shall call positive, i.e., so that the point which they include remains to the left; we shall denote with $-\ell_i$, $-\ell_x$ the same lines traversed in the opposite direction. Finally, $\ell_i + \ell_j$ will indicate the line formed by ℓ_i and ℓ_j successively traversed; it should be noted here that, although the addition sign is used, the commutative law cannot be regarded as applicable.

Having thus established this, we can assume as line λ satisfying the conditions of Sec. 42 the line

$$l_{hx} = l_h + l_x - l_h - l_x.$$

Indeed, after having traversed ℓ_h, the function under the sign of (32) is multiplied by $e^{2\pi i\alpha_h}$; after having traversed $\ell_h + \ell_x$, the factor $e^{-2\pi i\sigma}$ is acquired, after $-\ell_h$, the factor $e^{-2\pi i\alpha_h}$, finally after $-\ell_x$ the factor $e^{2\pi i\sigma}$, so that the function regains the primitive value. The same being the case for all its derivatives, and the definite integral also having meaning and the integration by parts and the derivation under the sign being clearly admissible, the required conditions for line λ are satisfied. We can also remove any doubt that the integral (32), extended to line ℓ_{hx}, is identically zero, with the exception of special cases; indeed, by denoting with I_h, I_x the integrals extended to ℓ_h, ℓ_x when the integral departs from t_o with a determinate value of the function under the sign, we readily find that

$$\int_{l_{hx}} = (1 - e^{-2\pi i\sigma})\, I_h - (1 - e^{2\pi i\alpha_h})\, I_x.$$

We can construct r, (h = 1, 2,... r) of such lines and no linear relation exists among the corresponding integrals because otherwise a linear dependence would result between

$$I_1, I_2, \ldots I_r, I_x$$

opposite the arbitrariness of the roots of $D_n(t)$.

It is evident that the conditions imposed on line λ can also be satisfied with line

$$l_{hk} = l_h + l_k - l_h - l_k$$

and thus $\int \ell_{hk}$ would be a new integral of the equation, which should therefore be linearly related to the preceding ones. Indeed it is evident that /270

$$\int_{hx} = (1 - e^{-2\pi i \sigma}) I_h - (1 - e^{2\pi i \alpha_h}) I_x ;$$

analogously

$$\int_{l_{hx}} = (1 - e^{-2\pi i \sigma}) I_k - (1 - e^{2\pi i \alpha_h}) I_x \quad , \quad \int_{l_{hk}} = (1 - e^{2\pi i \alpha_k}) I_y - (1 - e^{2\pi i \alpha_h}) I_k ,$$

from which it follows identically

$$(1 - e^{2\pi i \alpha_h}) \int_{l_h} = (1 - e^{2\pi i \alpha_h}) \int_{l_{kx}} + (1 - e^{-2\pi i \sigma}) \int_{l_{hk}} .$$

Under the condition that the integrals have meaning, we can substitute in the indicated integration lines, the line (which might also be straight) connecting a_h with a_k, or a_h with x, or a_h with ∞, or x with ∞; lines which can have practical advantage, but theoretically they have no advantage over those generally considered.

46. We must now indicate how it is possible to obtain the group of the equation (31). It will suffice therefore to know the manner in which the integrals extended to the lines ℓ_{hx} are transformed when the x value, considered now as a variable, rotates around one of the singular points. However, since the line ℓ_{hx} is formed from the simple lines ℓ_h, ℓ_x, it will suffice to show

how these should be transformed so that x can rotate around a_h without violating the exceptions established for the integration lines. It is clear that this rotation

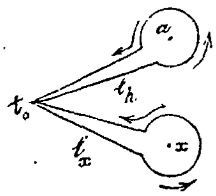


fig. 1.

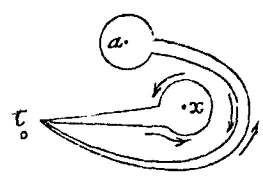


fig. 2.

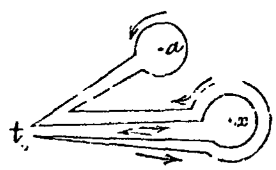


fig. 3.

will be possible if we substitute the ℓ_h' of Figure 2 for the ℓ_h of Figure 1, then the ℓ_x' of Figure 4 for ℓ_x. However these lines can be restored to the primitive lines by noting that for Figure 3, /271

$$l_h' = l_x + l_h - l_x .$$

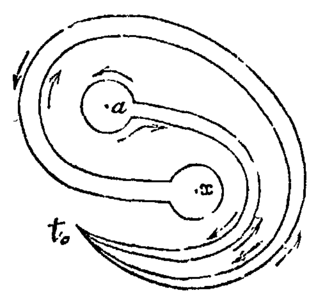


fig. 4.

and

$$- l_h' = l_x - l_h - l_x ,$$

and analogously

$$l_h' = l_h' + l_h - l_h'$$

then substituting:

$$l_x' = l_x + l_h + l_x - l_h - l_x ,$$

a relation which enables us to find immediately the substitution undergone by the integrals $\int \ell_{hx}$, $\int \ell_{hx}$ when x rotates around a_h.

47. The following observations can be added on the integrals of equation (31):

(a) Supposing that the line λ does not pass through the point a_h, let us consider the values of x for which it is

$$|x - a_h| < |t - a_h| ,$$

t being any point of the integration line. For such values of x, the binomial $(t-x)^{-\sigma+n-1}$ which appears under the [illegible]n in (32), can be expanded in power series of x - a_h, and substituting in (32) and integrating term by term, an expansion is obtained for $\psi(x)$ in power series of x - a_h, the coefficients of which, except for some numerical factors, are definite integrals of the same form as (32) but with one less binomial factor. Called the [illegible] hypergeometric function of order r by Pochhammer, it follows, by the same author, that:

"The coefficients of the expansion in series of a hypergeometric function of order r are hypergeometric functions of order r - 1"

(b) As stated in Sec. 43, a linear relation with rational coefficients /272 exists among n + 1 integrals (32) in which the values of σ differ by integers. The same occurs if any other of the exponents of the binomials of (32), e.g., a_h, is considered instead of σ. This can also be verified by direct calculation; indeed, by considering a_h as a variable, ψ satisfies, with respect to this, a linear differential equation of order n analogous to (31), since a_h enters under the sign in the same way as x; now we have

$$\frac{\partial\psi}{\partial a_h} = -\alpha_h\,\psi(\alpha_h - 1)$$

whence

$$\frac{\partial^k\psi}{\partial a_h{}^k} = (-1)^k\,\alpha_h{}^k\,\psi(\alpha_h - k)\,;$$

by substituting in the differential equation under discussion, this is transformed into a recurrent equation, with rational coefficients, among

$$\psi(\alpha_h)\quad,\quad\psi(\alpha_h - 1)\;.\;.\;.\;.\;\psi(\alpha_h - n).$$

(c) When applying that which has been said in the preceding Sections to the case of n = 2, and setting $a_1 = 0$ and $a_2 = 1$, which does not constitute a restriction, Δ becomes

$$(t^2 - t)\,\varphi'' + (at + b)\,\varphi' = 0$$

from which

$$\frac{\varphi''}{\varphi'} = \frac{b}{t} - \frac{a+b}{t-1}\quad.\quad\varphi' = c\,t^b\,(t-1)^{-(a+b)}\,;$$

equation (31) is thereby reduced to

$$(t^2 - t)\,\psi'' + \big(at + b + (2t-1)\,\sigma\big)\,\psi' + \big(a\sigma + \sigma(\sigma - 1)\big)\,\psi = 0\,,$$

which is integrated by

$$\psi = c\int_{(\lambda)} t^b\,(t-1)^{-(a+b)}\,(t-\omega)^{-\sigma}\,dt\,,$$

where the line λ is a closed line which encircles the points x and 0 and the points x and 1 twice, and can be reduced, if the exponents have their real part greater than -t, to the line which connects o with x, o with 1, 1 with x and also one of these points with ∞, if the sum of the exponents has the real part smaller than one. It is sufficient to set /273

$$a = \beta - \alpha + 1 \;, \quad b = \alpha - \gamma \;, \quad c = \alpha$$

in order to restore the differential equation to the form of the hypergeometric equation, which is integrated by

$$y(x) = c\int_{(l)} t^{\alpha-\gamma}(t-1)^{\gamma-\beta-1}(t-x)^{-\alpha}\,dt$$

which, by setting t = 1/n, is reduced to the definite integral cited in Sec. 5.

48. The equation (31) which we have studied presents the first generalization of the hypergeometric differential equation, known under the name of Pochhammer's generalization. Another generalization, credited to Goursat* is offered by a regular equation of the n^{th} order, having only the singular points x = 0, x = 1, x = ∞, of the form

$$(x^n - x^{n-1})y^{(n)} + (a_{n-1}x^{n-1} + b_{n-1}x^{n-2})y^{(n-1)} + \dots + (a_1x + b_1)y' + a_0y = 0. \tag{33}$$

It is easy to calculate the determinant equations relative to the three singular points, and they are, respectively:

for x = 0:

$$\rho(\rho-1)\dots(\rho-n+1) - b_{n-1}\rho(\rho-1)\dots(\rho-n+2) - b_{n-2}\rho(\rho-1)\dots(\rho-n+3) - \dots - b_1\rho = 0\,,$$

for x = 1

$$\rho(\rho-1)\dots(\rho-n+2)\{\rho-n+1+a_1+b_1\} = 0\,,$$

for x = ∞:

$$\rho(\rho-1)\dots(\rho-n+1) + a_{n-1}\rho(\rho-1)\dots(\rho-n+2) + \dots + a_1\rho + a_0 = 0\,;$$

*Annales de l'Ecole Normale, loc. cit.

thus it follows that there is one integral without singularity in the neighborhood of $x = 0$, $n - 1$ integrals without singularity in the neighborhood of $x = 1$, none (except for special values of the coefficients) in the neighborhood of $x = \infty$.

In determining the coefficients of the expansion in series of the integrals in the neighborhood of the points 0, 1 and ∞, we readily find that:

"The ratio between a coefficient and the preceding one is a fractional /274 rational function of the index, the rank of the terms of the fraction being n, as it is 2 for the Gauss series. Reciprocally, a power series in which the ratio between one coefficient and the preceding one is a rational function of the index satisfies one of Goursat's differential equations*."

49. One of the most remarkable observations made on the Goursat equation is that its integrals can be presented in the form of multiple definite integrals**. In the present section, we propose to obtain this result on the basis of the same observation which we used for the preceding generalization; i.e., by showing how all the equations included in the notation $S_\sigma \Delta = 0$*** are integrated immediately by means of the expression (29), once the regular equation $\Delta = 0$ has been integrated.

By applying the transformation S_σ to the equation (33), an equation in ϕ is obtained, the form of which is precisely the same as the primitive equation, which contains however the quantity σ rationally in its coefficients. Then, by choosing σ so that the coefficient of the last term is annulled, then setting

*Goursat, loc. cit.

**Pochhammer, Crelle, Vol. CII

***cf. Rendiconti della R. Accad. dei Lincei, May 1892.

$$\frac{d\varphi}{dx} = \theta ,$$

we have to integrate an equation of the form

$$(x^n - x^{n-1})\theta^{(n-1)} + (a'_{n-1}x^{n-1} + b'_{n-1}x^{n-2})\theta^{(n-2)} + \dots + (a_1'x + b_1')\theta = 0. \tag{34}$$

In this equation we set

$$\theta = x^{\lambda}\theta_1$$

and by dividing the resulting equation by x^{λ}, an equation of the same form as (34) is obtained

$$(x^n - x^{n-1})\theta_1^{(n-1)} + (a''_{n-1}x^{n-1} + b''_{n-1}x^{n-2})\theta_1^{(n-2)} + \dots + (a_1''x + b_1'')\theta_1 = 0 ,$$

whose coefficients however contain λ rationally. Now by choosing λ so that b_1'' becomes equal to zero, then dividing the equation by x, we have an equation of the primitive form (33), but whose order is reduced by one unit. By re- /275 applying the same procedure to this equation, and so on, we finally arrive at an equation of the second order which is the usual hypergeometric equation and whose integral can be put in the form of a definite integral, so that the integral of (33) can be put in the form of a definite integral of $(n-2)^{th}$ rank.

CHAPTER VII

RECURRENT EQUATIONS WITH RATIONAL COEFFICIENTS AND THEIR VARIOUS APPLICATIONS. SPHERICAL FUNCTIONS

50. In this chapter we propose to study the properties of the integrals of recurrent equations of the second order, the coefficients of which are rational functions of the index. This type of study, which is presented as an obvious application of the results of the preceding chapters, is of special interest because we can easily deduce from it the properties of the more well-

known and more frequently used recurring systems such as spherical functions, Jacob's polynomials, etc., especially with respect to the conditions for the expansion of a given analytic function, in ordinate series of the functions of these systems. We are limiting our treatment to the case of recurrent equations of the second order because the systems entering into ordinary applications are of this order, however, the extension to systems of higher orders presents no difficulties*.

Let us consider the recurrent (or difference) equation of the second order

$$a(n) f_{n+2} + b(n) f_{n+1} + c(n) f_n = 0 \tag{35}$$

where a(n), b(n) and c(n) are integers and rational polynomials of the same rank m with respect to the index n; by substituting the factorials** for the powers in these polynomials, and designating h(h - 1) .. (h - k + 1) with $(h)_k$, we shall write

$$a(n) = a_m (n+2)_m + a_{m-1} (n+2)_{m-1} + \dots + a_1 (n+2)_1 + a_0 ,$$
$$b(n) = b_m (n+1)_m + b_{m-1} (n+1)_{m-1} + \dots + b_1 (n+1)_1 + b_0 ,$$
$$c(n) = c_m (n)_m + c_{m-1} (n)_{m-1} + \dots + c_1 (n)_1 + c_0 ,$$

where a_m, a_{m-1}, ... a_0, b_m, ... b_0, c_m, ... c_0 are constants with respect to n, and $a_m c_m$ are assumed to be different than zero. /276

Together with this equation, we shall consider the following linear differential form

$$\Delta\varphi = (a_m + b_m t + c_m t^2) t^m \varphi^{(m)} + (a_{m-1} + b_{m-1} t + c_{m-1} t^2) t^{m-1} \varphi^{(m-1)} + \dots + (a_0 + b_0 t + c_0 t^2)\varphi ,$$

*See my memorandum: On the Generation of Recurrent Systems, etc. Acta Mathematica, Vol. XVI, 1892.

**See p. e. Capelli, Algebraic Analyses, etc., this Journal, Vol. XXXI (Sec. VII, 3).

and the non-homogeneous equation

$$\Delta\varphi = t^{\mu}(h + kt) \tag{36}$$

where μ is a positive integer, indeterminate for the present, and h and k are two constants.

The equation $\Delta\phi = 0$ is a regular homogeneous equation, whose singular points are $\underline{t} = 0$, $\underline{t} = \infty$, and the roots α, β of the equation of the second rank

$$a_m + b_m t + c_m t^2 = 0\,; \tag{37}$$

in regard to equation (36), its general integral is obtained by adding the general integral of $\Delta\phi = 0$ to one of its own particular integrals.

Now, equation (36) has a special integral, and generally only one, which can be expanded in the neighborhood of the point $\underline{t} = 0$ in positive whole number power series of $\underline{t}$. Writing this integral in the form

$$\varphi(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_n t^n + \ldots, \tag{38}$$

we immediately find that the coefficients p_0, $p_1, \ldots$ $p_{\mu-1}$ are zero, while for the others, the following equations hold:

$$\left\{\begin{array}{l} a(\mu-2)\,p_\mu = h\,, \\ a(\mu-1)\,p_{\mu+1} + b(\mu-1)\,p_\mu = k\,, \end{array}\right. \tag{39}$$

$$a(n)\,p_{n+2} + b(n)\,p_{n+1} + c(n)\,p_n = 0\,, \quad (n = \mu+1\,,\, \mu+2\,, \ldots).$$

The system of coefficients p_n of (38) is thus an integral of the given recurrent equation (35), uniquely determined from the conditions (39), namely, by the coefficients $\underline{h}$ and $\underline{k}$.

51. It is not difficult to recognize the radius of the circle of convergence of the series (38) by means of Poincaré's theorem given in Sec. 16. In fact, this radius is the inverse of the absolute value of the limit (if it ex-

ists) of the ratio $p_{n+1}:p_n$ when $\underline{n} = \infty$; now, since we have /277

$$\lim_{n=\infty} \frac{b(n)}{a(n)} = \frac{b_m}{a_m} \quad , \quad \lim_{n=\infty} \frac{c(n)}{a(n)} = \frac{c_m}{a_m} ,$$

the limit of $p_{n+1}:p_n$ will exist and will be, for the cited theorem, one of the roots of the equation (the reciprocal of (37))

$$a_m X^2 + b_m X + c_m = 0 ,$$

and in general it will be that root whose modulus is the greatest. Then supposing $|\alpha| < |\beta|$, we have

$$\text{generally} \quad \lim_{n=\infty} \frac{p_{n+1}}{p_n} = \frac{1}{\alpha} \quad , \qquad \text{exceptionally} \quad \lim_{n=\infty} \frac{p_{n+1}}{p_n} = \frac{1}{\beta} .$$

This is in relation with the observation that the singular points of the integral of (36) are $\underline{t} = \alpha$, $\underline{t} = \beta$, and one of these points (generally α) should lie on the circumference which limits the circle of convergence of (38).

52. We shall now recall how it follows from the principles of Chapters II and III that

(a) "If two integrals p_n and p'_n of the recurrent equation (36) are such that the limit of the ratio $p'_n : p_n$ is zero when $n = \infty$, p'_n will be the distinct integral of the same equation, and the continuous fraction which this equation defines will be convergent."

(b) "If two integrals of equation (35) are found so that for one, p_n, we have $\lim_{n=\infty} \frac{p_{n+1}}{p_n} = \frac{1}{\alpha}$ and for the other p'_n, $\lim_{n=\infty} \frac{p'_{n+1}}{p_n} = \frac{1}{\beta}$, the limit of $p'_n:p_n$ will be zero and p'_n will be the distinct integral."

Having established this, and by assuming that we have already found the distinct integral p'_n mentioned in (b), the determination of the σ value of the continuous fraction defined by (35) can be conveniently performed with the method in Sec. 15. Having set for brevity

$$-\frac{b(n)}{a(n)} = r_n \quad , \quad -\frac{c(n)}{a(n)} = s_n ,$$

equation (36) is written

$$f_{n+2} = r_n f_{n+1} + s_n f_n ;$$

this is satisfied by the numerators A_n and by the denominators B_n of the reductions of the continuous fraction /278

$$\cfrac{s_\mu}{r_\mu + \cfrac{s_{\mu+1}}{r_{\mu+1} + \cfrac{s_{\mu+2}}{r_{\mu+2} + \cdots}}} \quad \text{or} \quad \cfrac{c(\mu)}{b(\mu) - \cfrac{a(\mu)\, c(\mu+1)}{b(\mu+1) - \cfrac{a(\mu+1)\, c(\mu+2)}{b(\mu+2) - \cdots}}} \tag{40}$$

for which have been set

$$A_\mu = 1 \quad , \quad A_{\mu+1} = 0 \quad , \quad B_\mu = 0 \quad , \quad B_{\mu+1} = 1 .$$

Then we have

$$p'_n = p'_\mu A_n + p'_{\mu+1} B_n ,$$

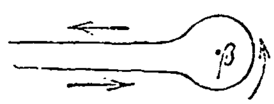
where, having assumed p'_μ to be different than zero (if it did equal zero, it would suffice to change μ into $\mu + 1$, etc.), divided by B_n, and taken the limit for $n = \infty$, we have

$$\sigma = \lim_{n=\infty} \frac{A_n}{B_n} = -\frac{p'_{\mu+1}}{p'_\mu} .$$

53. The only remaining requirement is that we find the distinct integral p'_n of (35) or, which is the same thing, the system of coefficients of the series expansion $\sum_{n=\mu}^{\infty} p'_n t^n$ which satisfies (36) and converges in a circle of center $\underline{t} = 0$ and of radius $|\beta|$; the ratio of the second of these coefficients to

the first, taken with the sign changed, will give the value of the continuous fraction defined by (35). This expansion is obtained by the method which follows:

We describe in plane t an indefinite line λ which, having left infinity, returns there in the same direction after having made a turn around point β, without having this line pass through the



point 0 nor the point α and without having this line contain either of these points in its interior (the region in which β lies). There is only one integral of the equation $\Delta\phi = 0$ which, by one turn of the variable around the point β, is reproduced multiplied by a constant: we denote this integral by U(t) /279
and we observe that it becomes infinite of necessarily finite order when t = ∞ by performing the analytic continuation along the line λ: let τ be the real part of this order of infinity. Finally, we determine the integer μ, hitherto indeterminate, so that it is:

$$\tau - \mu + 2 < 0.$$

With these positions, the definite integral

$$\varphi_1(z) = z^\mu \int_{(\lambda)} \frac{U(t)\,dt}{t^\mu (t-z)}$$

will have a meaning for each value of z external to the line λ; however, a calculation analogous to that performed in Sec. 42* readily demonstrates that $\phi_1(z)$ satisfies an equation

*For carrying on the calculation, see my cited memorandum in Acta Mathematica, Vol. XVI, Sections 5-9.

$$\Delta\varphi_1 = z^\mu (h + kz)$$

of the same form as (36), where

$$h = -c(\mu-2)\int_{(\lambda)} \frac{U(t)\,dt}{t^{\mu-1}} - b(\mu-2)\int_{(\lambda)} \frac{U(t)\,dt}{t^\mu}$$

$$k = -c(\mu-1)\int_{(\lambda)} \frac{U(t)\,dt}{t^\mu}$$

where both of the definite integrals have meaning because of the hypothesis done on μ.

Now the integral $\phi_1(z)$ can be expanded in power series of z,

$$\varphi_1(z) = \sum_{n=\mu}^{\infty} C_n z^\mu \quad , \quad C_n = \int_{(\lambda)} \frac{U(t)\,dt}{t^{\mu+n+1}} \tag{41}$$

convergent for all values of $|z| < |t|$; however, inasmuch as the line λ can be taken to be as close to β as desired, the circle of convergence of (41) has 0 for its center and $|\beta|$ for its radius. The system C_n is then the distinct integral of (35), which we wished to find.

It should be noted that the definite integrals which figure in the expressions of h and k can be represented with $C_{\mu-2}$ and $C_{\mu-1}$; upon substituting the values of h and k in the expressions (39), these assume the form of equation (35) for the values $n = \mu - 2$, $n = \mu - 1$. /280

Summarizing:

"Given the recurrent equation (35), the coefficients of which are integers and rational polynomials of the same rank n, the system C_n given by the formula (41) is the distinct integral of it. Then forming the continuous fraction (40), the numerators and denominators of which satisfy (35), it is convergent and its values are given by the ratio $-C_{\mu+1}:C_\mu$*."

It should be observed that in the preceding definite integrals, a line which goes from β to infinity can be substituted for the integration line λ, when the function under the sign for $\ell = \beta$ is infinite of such an order that its real part is less than 1. Then, the integrals $\int_\beta^\infty$, which do not differ except by a constant factor, can be substituted for the integrals $\int_\lambda$.

54. The preceding result, namely, the method of expressing the value of each continuous fraction of the form (40) as a ratio of two definite integrals** includes as a special case the well-known Gauss formula*** which gives the expansion in the continuous fraction of the ratio of two hypergeometric functions. In order to obtain this formula, it suffices to reduce the form $\Delta\phi$ to the first order, assuming all the a_h, b_h and c_h values from the index (subscript) $h = m$ to $h = 2$ inclusively to be zero; in this case the continuous fraction (40) takes the form

$$\cfrac{c_1\mu + c_0}{b_1(\mu+1) + b_0 - \cfrac{(a_1(\mu+2)+a_0)(c_1(\mu+1)+c_0)}{b_1(\mu+2)+b_0 - \cfrac{(a_1(\mu+2)+a_0)(c_1(\mu+2)+c_0)}{b_1(\mu+3)+b_0 - \dots}}}$$

the differential equation $\Delta\phi = 0$ becomes

$$(a_1 + b_1 l + c_1 l^2)\, l\varphi' + (a_0 + b_0 l + c_0 l^2)\,\varphi = 0\,.$$

*It remains to consider the convergence of the continuous fraction and the search for its value in the case in which $a_m = 0$, and in that in which $|\alpha| = |\beta|$: we leave this easy determination to the reader.

**See Rendiconti della R. Accademia dei Lincei, June 21, 1891.

***Werke, Vol. III, p. 134. Cf. Heine, Handbook of Spherical Functions, Vol. I, pp. 269 and 280.

the integral of which is of the form $t^{\xi}(t-\alpha)^{\eta}(t-\beta)^{\zeta}$, and the value of the continuous fraction is /281

$$-\int_{(\lambda)} t^{\xi-\mu-1}(t-\alpha)^{\eta}(t-\beta)^{\zeta}\,dt : \int_{(\lambda)} t^{\xi-\mu-2}(t-\alpha)^{\eta}(t-\beta)^{\zeta}\,dt,$$

which can be easily expressed (Sec. 47, c) by means of the quotients of two hypergeometric series.

---

55. In the preceding Sections the coefficients a(n), b(n), c(n) of the equation (35) were assumed to be dependent only on the index n. We suppose now that b(n) contains linearly a parameter x and is

$$(u)_{,,}q + x\,(u)_{,}q = (u)\,q$$

with

$$b'(n) = b'_m(n+1)_m + b'_m\ (n+1)_{m-1} + \ldots + b'_0,$$

$$b''(n) = b_m''(n+1)_m + b_{m-1}''(n+1)_{m-1} + \ldots + b_0'';$$

so that the equation which we shall consider will be

$$a(n)\,f_{n+2} + \big(b'(n)\,x + b''(n)\big)\,f_{n+1} + c(n)\,f_n = 0. \tag{42}$$

The integrals of equation (42) will be functions of x, we shall be able to give particular consideration to the integral $B_n(x)$ defined by the initial conditions $B_\mu(x) = 0$, $B_{\mu+1}(x) = 1$, and which will be composed by a system of whole number rational polynomials in x of rank ordinately increasing by one unit; precisely, $B_n(x)$ is of the rank $n - \mu - 1$.

As in Sec. 50, the linear differential form $\Delta\phi$ can be made to correspond to the equation (35), and the singular points of $\Delta\phi = 0$ are the roots of the equation

$$a_m + (b_m' x + b_m'') X + c_m X^2 = 0 ;$$

let $\alpha(x)$ and $\beta(x)$ be the roots of this equation, and be $|\alpha(x)| < |\beta(x)|$. Each integral f_n of the recurrent equation (42), with the exception of the distinct integral, in such that

$$\lim_{n=\infty} \frac{f_{n+1}}{f_n} = \frac{1}{\alpha(x)};$$

in particular, this will usually occur for the integral $B_n(x)$; (in the event that $B_n(x)$ coincides with the distinct integral, another integral, e.g., the A_n integral defined by $A_\mu = 1$, $A_{\mu+1} = 0$, would be substituted for $B_n(x)$).

Equation (42) belongs to the type of equation (5') studied in Sec. 21; /282 together with this equation, we shall consider its inverse equation, as in that Section

$$a(n-1) f_n + \big(b'(n) z + b''(n)\big) f_{n+1} + c(n+1) f_{n+2} = 0. \tag{43}$$

This also is of the form of (35): therefore, a linear differential form $\Delta_1\phi$ will also correspond to it, and the equation $\Delta_1\phi = 0$ will have for singular points, in addition to 0 and ∞, the roots of the equation

$$c_m + (b_m' z + b_m'') X + a_m X^2 = 0 ,$$

which are $1/\alpha(z)$ and $1/\beta(z)$. The theory, established in Sec. 52 and in subsequent ones, indicates how the distinct integral of equation (43) is determined; indicating this integral with $S_n(z)$, we will have

$$\lim_{n=\infty} \frac{S_{n+1}(z)}{S_n(z)} = \alpha(z) ,$$

and for the value $n = \mu - 1$, equation (43) relative to S_n gives

$$c(\mu)\, S_{\mu+1} + (b'(\mu-1) z + b''(\mu-1))\, S_\mu = k\,,$$

where the determination of k proceeds in accordance with Sec. 53.

Let us now take up the development (7) given in Sec. 21, in which we change n into $n + \mu$:

$$\frac{1}{k} \sum_{n=\mu+1}^{\infty} b'(n-1)\, B_n(x)\, S_n(z);$$

let us also take x so that $|\alpha(x)| > \rho + \varepsilon$, and z so that $|\alpha(z)| < \rho - \varepsilon$, ρ being an arbitrary positive quantity and $\varepsilon < \rho$ an arbitrarily small positive quantity; the expansion will be convergent absolutely and in equal rank, and then upon repeating on it the formal calculation performed in Sec. 21, it will follow that the sum of this expansion will be $1/(z - x)$, and thus we have

$$\frac{1}{z-x} = \frac{1}{k} \sum_{n=\mu+1}^{\infty} b'(n-1)\, B_n(x)\, S_n(z) \tag{7'}$$

under the condition $|\alpha(x)| > |\alpha(z)|$.

56. We denote with Γ_ρ the locus of points of the plane x for which we have $|\alpha(x)| = \rho$. This locus will generally be a curve which will separate the /283 region E'_ρ of the plane in which $|\alpha(x)| < \rho$ from that E_ρ in which $|\alpha(x)| > \rho$; it is clear that no passage can be made from one to the other without traversing Γ_ρ. The curves Γ_ρ limit the areas of convergence of the series of functions $B_n(x)$ or $S_n(x)$. Two curves, Γ_ρ Γ_{ρ_1}, cannot be cut off and if $\rho_1 < \rho$, Γ_{ρ_1} will be entirely in E'_ρ.

57. Now let $f(x)$ be an analytic and single valued function of x, given in the internal region of E_ρ; we shall have by the well-known Cauchy theorem:

$$f(x) = \frac{1}{2\pi i} \int_{(\Gamma_{\rho_1})} \frac{f(z)\,dz}{z - x},$$

$\underline{z}$ being taken outside of the region E_ρ, i.e., in the region E_ρ', and Γ_{ρ_1} being taken so that it is entirely in E_ρ', i.e., having $\rho_1 < \rho$. The series of (7') being convergent uniformly for the z values of Γ_{ρ_1} and for $\underline{x}$ within E_ρ, its expansion can be substituted for $1/z-x$ and thus we have the expansion of the function given $f(x)$ in series of the functions $B_n(x)$, and convergent in the entire E_ρ:

$$f(x) = \Sigma\, C_n B_n(x), \quad \text{con} \quad C_n = \frac{b'(n-1)}{2\pi i} \int_{(\Gamma_1)} \frac{S_n(z) f(z)\,dz}{k},$$

Analogously we can have the expansion of an analytic function given in the region E_ρ', in series of the functions $S_n(z)$.

---

58. The preceding considerations can be readily applied to the study of special recurrent systems with interesting results. We shall limit ourselves to giving an example of these applications, demonstrating how our method treats easily the study of spherical functions or Legendre polynomials and of expansion of analytic functions in series of such polynomials. The special case of spherical functions is presented when we put in equation (42)

$$a(n) = n + 2\ , \ b'(n) = -(2n + 3)\ , \ b''(n) = 0\ , \ c(n) = n + 1\ ;$$

upon writing the special equation which is obtained with these positions, we have

$$(n + 2) f_{n+2} - (2n + 3)\, x f_{n+1} + (n + 1) f_n = 0. \tag{44}$$

This equation does not differ from its inverse, except by the change of /284 x into z; therefore, in (7') the system S_n will be the distinct integral of (44) itself.

We shall consider the integral $B_n(x)$ of (44), defined by $B_0 = 0$, $B_1 = 1$; upon changing <u>n</u> into <u>n</u> - 1 in (44), and setting $P_n(x) = B_{n+1}(x)$, the recurrent equation becomes

$$(n+1) f_{n+1} - (2n+1) x f_n + n f_{n-1} = 0, \tag{44'}$$

and its integral $P_n(x)$ is constituted by the system

$$P_{-1}(x) = 0 \ , \ P_0(x) = 1 \ , \ P_1(x) = x \ , \ P_2(x) = \frac{3}{2}\left(x^2 - \frac{1}{3}\right), \ \ldots$$

of polynomials of rank equal to their index. These polynomials are known under the name of Legendre <u>polynomials</u> or <u>spherical functions of the first order</u>. They are the denominators of the reduced of the continuous fraction

$$\cfrac{1}{x - \cfrac{\frac{1}{2}}{\frac{3}{2}x - \cfrac{\frac{2}{3}}{\frac{5}{3}x - \cdots}}} \qquad \text{or} \qquad \cfrac{1}{x - \cfrac{1\cdot 1}{3x - \cfrac{2\cdot 2}{5x - \cfrac{3\cdot 3}{7x - \cdots}}}}$$

defined by the recurrent equation (44').

It follows from the general theory (Sec. 50) that if f_n is an integral of (44'), the expansion $\Sigma f_n t^n$ is an integral of the linear differential equation

$$(1 - 2tx + t^2) t\varphi' - (xt - t^2)\varphi = h + kt; \tag{45}$$

taking for f_n the system $P_n(x)$, it is readily evident that <u>h</u> = <u>k</u> = 0; therefore, the series $\sum_{n=0}^{\infty} P_n(x) t^n$ satisfies the equation

$$(1 - 2xt + t^2)\varphi' - (x - t)\varphi = 0$$

which, integrated, gives

$$\varphi = C(1 - 2xt + t^2)^{-\frac{1}{2}},$$

however, C = 1 because $P_0 = 1$, and therefore /285

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = (1 - 2xt + t^2)^{-\frac{1}{2}}.$$

Thus, the property that usually defines the polynomials P_n is found again; they are, namely, the coefficients of the expansion of $(1 - 2xt + t^2)^{-1/2}$ in power series of t.

On the basis of this definition, it is not difficult to obtain the expansion of $P_n(x)$, from which it is evident that, with the exception of a numerical factor, $P_n(x)$ is a hypergeometric series (reduced to a polynomial); and precisely, P_{2n} coincides with $F(-n, n + 1/2, 1/2, x^2)$ and P_{2n+1} with $xF(-n, n + 3/2, 3/2, x^2)$, with the exception of a numerical multiplication factor*.

59. The singular points of equation (45), in addition to t = 0 and t = ∞, are the roots of the equation of second rank in t.

$$1 - 2tx + t^2 = 0. \tag{46}$$

With the exception of the real values of x included between -1 and +1, for which the two roots of (46) both have the modulus equal to unity, one of the roots of this equation has a modulus greater than unity, while the other is the reciprocal of the first and, therefore, has its modulus less than unity. We shall denote the first with $\beta = r(x)$, the other will be $\alpha = 1/4(x)$. The locus of points of the plane x for which $|r(x)|$ has a constant value in an ellipse having the points +1 and -1 as foci; as $|r(x)|$ increases from 1 to ∞, the elli-

*For the other elementary properties of the functions P_n, see the first Chapter of the frequently cited Handbook by Heine.

pse increases in size from the segment -1.. +1 to an ellipse of infinite axes. These same ellipses are also the loci in which the modulus of that root α with modulus less than unity remains constant; precisely, the curve Γ_ρ (Sec. 56) on which we still have

$$|\alpha| = \rho, \qquad \text{whence} \quad |r(x)| = \frac{1}{\rho}, \quad (\rho < 1)$$

is the ellipse represented by the equation, having set x = u + iv:

$$\frac{u^2}{\frac{1}{4}\left(\frac{1}{\rho}+\rho\right)^2} + \frac{v^2}{\frac{1}{4}\left(\frac{1}{\rho}-\rho\right)^2} = 1,$$

and the region E_ρ is the internal part and the region E'_ρ the external part of the ellipse Γ_ρ.

60. We now turn our attention to determining the distinct integral of /286 equation (44). This is obtained by applying the general method given in the present Chapter. If necessary, we again take up the definite integral

$$\int \frac{U(t)\,dt}{t^\mu(t-\zeta)}$$

considered in Sec. 53; here we shall have $\mu = 0$, $U(t) = (t^2 - 2tx + 1)^{-1/2}$, and line λ surrounds the root of equation (46) which has the larger modulus namely r(x). Inasmuch as U(t) is infinite of order < 1 for t = r(x), we can substitute

$$J = \int_{r(x)}^{\infty} \frac{dt}{(t-\zeta)\sqrt{t^2 - 2tx + 1}}$$

for the preceding integral and, expanding this integral in power series of ζ, the coefficients of the successive powers of ζ will give the distinct integral of (46).

We shall set

$$J = \sum_{n=0}^{\infty} Q_n \zeta^n , \qquad \text{with } Q_n(x) = \int_{r(x)}^{\infty} \frac{dt}{t^{n+1}\sqrt{t^2 - 2tx + 1}} . \tag{47}$$

It is worthwhile to observe that, having fixed ζ, the expansion of J converges for all x values so that $|r(x)| > |\zeta|$, i.e., <u>outside</u> of one of the ellipses Γ_ρ if $|\zeta| = \rho > 1$, and in the entire plane except the segment -1.. 1, if $|\zeta| < 1$. $Q_n(x)$ have been given the name of <u>spherical functions of the second order</u>.

We know that for the distinct integral the recurrent equation, in our case (44), is not valid for the initial values n = 0, n = 1, i.e., equation (44') is not valid for n = -1, n = 0. Then we calculate $Q_0(x)$ and $Q_1(x)$ directly. We have

$$Q_0(x) = \int_{r(x)}^{\infty} \frac{dt}{t\sqrt{t^2 - 2tx + 1}} .$$

Setting

$$\sqrt{t^2 - 2tx + 1} = u - t ,$$

we obtain, noting that $r(x) = x + \sqrt{x^2 - 1}$,

$$Q_0(x) = \frac{1}{2} \log \frac{x+1}{x-1} ;$$

with the same position

$$Q_1(x) = \int_{r(x)}^{\infty} \frac{dt}{t^2 \sqrt{t^2 - 2tx + 1}}$$

becomes

$$Q_1(x) = 4 \int_{r(x)}^{\infty} \frac{(u - x)\, du}{(u^2 - 1)^2}$$

and dividing

$$\int \frac{du}{(u^2-1)^2} \quad \text{into} \quad \int \frac{u^2\,du}{(u^2-1)^2} - \int \frac{du}{u^2-1}$$

and integrating the first by parts and limiting, we obtain

$$Q_1(x) = x\,Q_0(x) - 1. \tag{48}$$

This relation will be the one to be used instead of (44') for x = 0.

Now we can apply the expansion (7') of Sec. 55 to the spherical functions and we have, since k = -1, b'(n) = -(2n + 3),

$$\frac{1}{z-x} = \sum_{n=0}^{\infty} (2n+1)\,P_n(x)\,Q_n(z)\,, \quad (*) \tag{49}$$

convergent in equal rank for the x values represented by the internal points and the z values by the external points of an ellipse of foci ± 1. This formula permits of expanding in series of $P_n(x)$, a given, analytic and single-valued function within an ellipse of this homofocal system, or in series of $Q_n(z)$, a given, analytic and single-valued function regularly outside of one of the above mentioned ellipses.

61. The value σ of the definite continuous fraction from (44')

$$\sigma = \cfrac{1}{x - \cfrac{1^2}{3x - \cfrac{2^2}{5x - \cfrac{3^2}{7x - \dots}}}}$$

can be easily found. The numerators of its reductions, which we shall in- /288 dicate with N_n, form a system of whole number rational polynomials which satisfy

---

*A formula often credited to Neumann, but which originates with Heine (Crelle, Vol. 42, p. 72). Cf. C. Neumann, On the Development of a Function with an Imaginary Argument, etc., (Halle, Schmidt, 1862) and Thomé, Crelle, Vol. 66.

(44') and, together with the denominators P_n, give a fundamental system of (44') itself. The integral N_n of (44') is defined by the initial conditions $N_{-1} = 1$, $N_0 = 0$. By denoting two constants with respect to th index n with c and c', we can set

$$Q_n = c\,N_n + c'\,P_n\,,$$

and making n = 0, we have $Q_0 = c'$; then making n = 1, ($N_1 = 1$, $P_1 = x$), we have

$$Q_1 = c + Q_0\,x$$

and comparing with (48), we have c = -1. Whence it follows, since

$$Q_0 = \frac{1}{2}\log\frac{1+x}{1-x},$$

the relation between the spherical functions of the first and second species:

$$Q_n = \frac{1}{2}P_n\log\frac{1+x}{1-x} - N_n\,, \tag{50}$$

and since

$$\lim_{n=\infty}\frac{N_n}{P_n} = \sigma \quad , \quad \lim_{n=\infty}\frac{Q_n}{P_n} = 0\,,$$

we have

$$\log\frac{1+x}{1-x} = \cfrac{1}{x - \cfrac{1^2}{3x - \cfrac{2^2}{5x - \dots}}} \qquad = \frac{1}{2}\int_{-1}^{1}\frac{dt}{t-x} \tag{51}$$

a formula by Gauss, and valid for every value of x except the real values between -1 and +1.

The formula (50) determines completely the nature of the spherical functions of the second order. Since log 1 + x/1 - x is a singular, (logarithmically) multiform and analytic function only at the points x = $\pm$ 1, and that P_n and N_n

are whole number rational functions, it follows that Q_n has the same singularities and, therefore, it is a multiform analytic function; however its branches can be separated by dividing the plane of the variable along the real axis between the points +1 and -1. Moreover, (Sec. 17) the function Q_n is zero to infinity of the order n + 1. /289

It is quite remarkable that while the expansion in power series of 1/x of the function 1/2 log 1 + x/1 - x is valid only outside of the circle of center x = 0 and of radius 1, the expansion (51) in a continuous fraction and the expression in the form of a definite integral are valid for the entire plane, with the exception of the segment -1 ... +1 of the real axis.

62. The relation (50) can also be written

$$Q_n(x) = \frac{1}{2} P_n(x) \int_{-1}^{1} \frac{dt}{t - x} - N_n(x)$$

or

$$Q_n(x) = -\frac{1}{2}\int_{-1}^{1} \frac{P_n(t) - P_n(x)}{t - x}\, dt + \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)\, dt}{t - x} - N_n(x);$$

it should now be noted that the first integral is a whole number rational function of x, because $P_n(t) - P_n(x)$ is divisible by t - x, while the second can be expanded in power series of 1/x, zero for x = ∞; we deduce from this that we should have separately

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)\, dt}{t - x} \tag{52}$$

and

$$N_n(x) = -\frac{1}{2}\int_{-1}^{1} \frac{P_n(t) - P_n(x)}{t - x}\, dt.$$

However $Q_n(x)$ is zero of the order n + 1 for x = ∞, the same should then hold for the second member and therefore we shall have

$$\int_{-1}^{1} P_n(t)\, t^k\, dt = 0 \qquad \text{per} \qquad k = 0, 1, 2, \ldots n \quad 1. \tag{53}$$

It results from this that

$$\int_{-1}^{1} P_n(t)\, R(t)\, dt$$

is zero for every whole number rational polynomial R(t) of an order less than n, and, in particular /290

$$\int_{-1}^{1} P_m(t)\, P_n(t)\, dt = 0 \tag{54}$$

provided that m $\gtrless$ n. We can readily deduce from this last property that the expansion of an analytic function in series of $P_n(x)$ can be performed in only one manner*.

63. $P_n(x)$, being, as stated, hypergeometric functions in which two parameters vary by integers an n varies, satisfy the linear differential equations having the same group. These equations could be obtained from the hypergeometric one but they can also be obtained by the following method. Having set:

$$R = \sqrt{t^2 - 2tx + 1}$$

we readily have

$$(1 - x^2)\frac{\partial^2 R}{\partial x^2} + t^2\frac{\partial^2 R}{\partial t^2} = 0\,;$$

deriving with respect to x, and noting that $\partial R/\partial x = t/K$, we have

$$(1 - x^2)\frac{\partial^2 \frac{1}{R}}{\partial x^2} - 2x\frac{\partial \frac{1}{R}}{\partial x} + t\frac{\partial^2 \frac{t}{R}}{\partial t^2} = 0\,;$$

*The properties given in this Section are the immediate consequences of the elementary theory of continuous algebraic fractions. See the indications given at the beginning of Sec. 21 for more information.

substituting for 1/R the expansion $\Sigma P_n(x)t^n$ and equating the coefficient of t^n to zero, we have the linear differential equation of the $P_n(x)$:

$$(1 - x^2)\frac{d^2 P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)P_n = 0. \tag{55}$$

Formula (52) immediately demonstrates (Sec. 42) that $Q_n(x)$ is a second integral of this equation and that, together with P_n, it gives the fundamental system of it.

64. The methods of the present Chapter, which we have applied to the /291 simple case of spherical functions, can be used with equal facility in the study of other more general polynomial systems. We cite, e.g., those which arise from the hypergeometric series in which the system of values -1, -2, -3,..., is substituted for one of the parameters α and β; this case includes the polynomial system first considered by Jacobi* and then studied by Darboux**. The properties of these polynomials, especially the possibility of expanding a given analytic function in series of them and the relative convergence conditions could be found quite readily by the methods indicated above, methods which adapt themselves to the study of recurrent systems of order greater than the second, as we find, for instance, in the reduction formulas of the hyperelliptic integrals.

August, 1894.

Translated for the National Aeronautics and Space Administration by the FRANK C. FARNHAM COMPANY

*Crelle, Vol. 56, p. 149.

**Journal de Mathematiques, S. III, Vol. IV, 1878.